AEDC-TR-74-79

cy. 1

DRAG FORCE ON OBJECTS IN THE NEARLY FREE MOLECULAR FLOW REGIME AS A FUNCTION OF SPEED RATIO

Y. Y. Lin Wang and J. V. Sengers

INSTITUTE FOR MOLECULAR PHYSICS
UNIVERSITY OF MARYLAND, COLLEGE PARK, MARYLAND 20742

September 1974

Final Report for Period February 15, 1973 — February 14, 1974

Approved for public release; distribution unlimited.

Prepared for

ARNOLD ENGINEERING DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
ARNOLD AIR FORCE STATION, TENNESSEE 37389

APPROVAL STATEMENT

This technical report has been reviewed and is approved.

Elton R. Thompson

Robert O Dietz

ELTON R. THOMPSON
Research and Development
Division
Directorate of Technology

ROBERT O. DIETZ
Director of Technology

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
AEDC-TR-74-79

2. GOVT ACCESSION NO.

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
DRAG FORCE ON OBJECTS IN THE NEARLY FREE MOLECULAR FLOW REGIME AS A FUNCTION OF SPEED RATIO

5. TYPE OF REPORT & PERIOD COVERED
Final Report-February 15, 1973 to February 14, 1974

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
Y. Y. Lin Wang and J. V. Sengers
University of Maryland
College Park, Maryland 20742

8. CONTRACT OR GRANT NUMBER(s)
F40600-73-C-0006

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Institute for Molecular Physics
University of Maryland
College Park, Maryland 20742

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Arnold Engineering Development Center (DYFS)
Arnold Air Force Station
Tennessee 37389

12. REPORT DATE
September 1974

13. NUMBER OF PAGES
97

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)

15. SECURITY CLASS. (of this report)
UNCLASSIFIED

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16 DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

1. Drag

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

2. Disc -- Drag
3 Spheres -- "

18. SUPPLEMENTARY NOTES

Available in DDC

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
drag coefficients
gaseous materials
aerodynamic drag
rarefied gases
free molecular flow

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The drag coefficient C_D of an object moving in a rarefied gas can be expanded in terms of the inverse Knudsen number K^{-1}: $C_D=C_o+C_1K^{-1}+\ldots$. The term C_o represents the drag coefficient in the free molecular flow limit. It is shown that the coefficient C_1 is determined by a set of well defined collision integrals associated with sequences of successive collisions among two molecules and the object. These collision integrals are evaluated for a disc and a sphere assuming that the gas molecules are

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

20. ABSTRACT (Continued)

reflected diffusively by the object. The molecules are treated as hard spheres and the drag coefficient is studied as a function of the velocity of the object.

AFSC
Arnold AFS Tenn

PREFACE

The research reported herein was sponsored by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC). The research was conducted at the Institute for Molecular Physics of the University of Maryland from February 15, 1973 to February 14, 1974 under delivery order F40600-73-0006. The Air Force project monitor for this project was Elton R. Thompson, AEDC (DYR). The manuscript was submitted for publication April 16, 1974.

The authors are indebted to Professor J. R. Dorfman, Dr. W. A. Kuperman and Dr. C. F. McClure for many valuable contributions to this research effort. In particular, the material presented in Chapter II is based on the earlier work documented in the Ph.D. dissertations of Dr. C. F. McClure and Dr. W. A. Kuperman. The authors also acknowledge some stimulating discussions with Ms. Y. H. Kan and Dr. A. G. Keel.

Computer time for this project was provided by the Computer Science Center of the University of Maryland.

The reproducibles used in the reproduction of this report were supplied by the authors.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont.)

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

This report is concerned with the problem of calculating the drag force on an object moving in a gaseous medium. An important parameter that governs the properties of aerodynamic flows in rarefied gases is the Knudsen number K. This Knudsen number is defined as the ratio of the mean free path λ of the molecules in the gas and a length L which measures the size of the object. In order to describe the aerodynamic processes when the Knudsen number is larger than unity, one needs to consider methods based on the kinetic theory of gases. That is, in rarefied gas dynamics, the drag force can be considered as resulting from collisions of individual molecules with the object.

The regime in which the molecular mean free path is much larger than the size of the object, i.e. the limit of infinite Knudsen number, is usually referred to as the free molecular flow regime. In this regime molecules that are reflected from the object collide on the average with oncoming molecules at large distances from the object. Hence the process is here completely determined by the interaction of independent gas molecules and the object, while collisions between the molecules may be neglected. Aside from the detailed nature of the interaction between gas molecules and the object, the physics of the free molecular flow regime

is well understood and practical calculations abound in the literature. Reviews of Schaaf [1] and Schaaf and Chambre [2] give examples of free molecular flow calculations together with bibliographies.

In this report we focus our attention on the nearly free molecular flow regime where the Knudsen number is greater than one, but definitely finite. For such flows the molecular mean free path is larger than a typical dimension of the object, but small enough so that collisions between the gas molecules cannot be neglected.

The drag coefficient C_D of an object in a gas stream is defined as

$$C_D = \frac{E}{U_K} \quad , \tag{1-1}$$

where E is the magnitude of the force exerted on the object and U_K is the incident kinetic energy. In the nearly free molecular flow regime $K^{-1} << 1$ the drag coefficient C_D may be written in the form

$$C_D = C_o + C_1 K^{-1} + \ . \ . \ . \quad , \tag{1-2}$$

where C_o is the drag coefficient in the free molecular flow regime $K^{-1} \to 0$ and C_1 the coefficient of a correction term which is inversely proportional to the Knudsen number K.

The drag force can be evaluated theoretically by solving the Boltzmann equation subject to the appropriate boundary conditions imposed by the presence of the object. The Boltzmann equation is a nonlinear integro-differential equation that describes the rate of change of the single-particle distribution function f of a dilute gas [3]

$$\frac{\partial f(\vec{r}_1, \vec{v}_1; t)}{\partial t} + \vec{v}_1 \cdot \frac{\partial f(\vec{r}_1, \vec{v}_1; t)}{\partial \vec{r}_1} = J(ff) \quad , \tag{1-3}$$

where $\vec{r}_1$, $\vec{v}_1$ represent the position and velocity of a molecule labeled 1

and where J(ff) is a collision term containing a time independent integral operator acting on the function $f(\vec{r}_1,\vec{v}_1;t)$. The term J(ff) accounts for the change in f due to binary collisions between the molecules and the gas. An approximate solution of the Boltzmann equation was obtained by Liu et al. for a sphere in a gas stream of Maxwellian molecules, i.e. molecules that repel each other with a force that varies as the inverse fifth power of the intermolecular separation [4].

Since it has proven to be difficult to solve the full nonlinear Boltzmann equation, some of the more authoritative studies are restricted to flows at low velocities where the linearized Boltzmann equation may be used [5]. Many authors have replaced the Boltzmann equation with a model equation proposed by Bhatnagar, Gross and Krook [6].

$$\frac{\partial f}{\partial t} + \vec{v}_1 \cdot \frac{\partial f}{\partial \vec{r}_1} = \nu(f_o - f) \quad , \qquad (1\text{-}4)$$

where f_o is the local Maxwell distribution and ν an adjustable parameter representing a velocity independent collision frequency. Willis has solved the BGK equation using a Knudsen iteration method [7]. Rose has used a Fourier transform technique to deduce the drag from the BGK equation [8]. Cercignani and coworkers have evaluated the drag of a sphere from the BGK equation using a variational method [9]. For a review of the use of the Boltzmann equation and the BGK equation in rarefied gas dynamics the reader is referred to the books of Kogan [10] and Cercignani [11]. Although judicious application of the BGK equation has yielded some encouraging results, the model equation does not have any really predictive power [12].

Another approach, encountered in the literature, is the first collision model. The first collision model assumes that the drag on the object is the sum of the free molecular flow drag C_o and a correction term which results from collisions between molecules reflected from the surface of the object and molecules in the incident beam. It was used by Lunc and Lubonski [13] for calculating the drag of a strip, by Baker and Charwat [14] for calculating the drag of a sphere, and by Perepukhov for calculating the drag of a sphere and a cone [15]. The first collision model is a phenomenological theory and it is not derived from first principles. Clearly as the Knudsen number decreases, more and more collisions between molecules must be considered.

A systematic approach to account for the effect of molecular collisions on the drag was developed by Dorfman and coworkers [16,17]. In this approach the object is treated as a heavy particle and the dynamical evolution of the system of gas and object is treated with the aid of the same techniques used earlier to derive the generalized Boltzmann equation for a moderately dense gas from the Liouville equation. In general the theory leads to a density expansion for the drag force on the object. In the limit that the mean free path of the molecules is large compared to the size of the molecules themselves, this expansion reduces to an expansion in the inverse Knudsen number. The coefficients in this expansion are given by integrals related to sequences of successive collisions among the molecules and the object. The same collision integrals can also be obtained by solving the Boltzmann equation with a modified Knudsen number iteration procedure as shown by Kelly and Sengers [18]. The derivation of the collision integrals for the first inverse

Knudsen number correction term in (1-2) is reviewed in Chapter II. A preliminary analysis of the collision sequences that enter into the evaluation of the coefficient C_1 in (1-2) was made by Kuperman [19].

In principle, our method of calculating the drag coefficient from collision integrals can be applied to objects of any shape in a gas stream of molecules with any interaction potential of finite range given the interaction mechanism between the molecules and the object. However, in our current studies of the nature of the collision integrals we have introduced the following approximations:

1). The molecules that strike the object do not stick to it, but are re-emitted after a time short compared to the mean free time of the molecules.

2). The molecules are re-emitted diffusively with a temperature T corresponding to the temperature of the object which is assumed to be the same as the temperature of the molecules in the gas stream. We are thus neglecting any heat transfer effects.

3). The molecules in the gas stream are assumed to interact as hard spheres with mass m and diameter σ.

The drag force on the object not only depends on the Knudsen number of the system, but also on the Mach number of the gas stream. The Mach number M is defined as the ratio of the flow velocity V relative to the sound velocity. For the problem at hand, instead of the Mach number, we find it more convenient to use as a parameter the speed ratio. The speed ratio S is defined as the ratio of the flow velocity V relative to the thermal velocity $(2kT/m)^{1/2}$, where k is Boltzmann's constant:

$$S = V(m/2kT)^{1/2}. \tag{1-5}$$

The speed ratio S is directly proportional to the Mach number M

$$S = M(\gamma/2)^{1/2} \quad , \qquad (1\text{-}6)$$

where γ is the specific heat ratio c_p/c_v [1].

In a preceding research effort Kuperman analyzed the collision integrals in the limits of zero and infinite Mach number [19]. The research effort reported here is concerned with a study of the feasibility of evaluating these collision integrals for all Mach numbers or speed ratios. In particular, we shall present results obtained for the drag coefficient as a function of the speed ratio for a disc and a sphere in Chapters III and IV, respectively. We were informed at the initial stage of this research effort that Willis had obtained from the Boltzmann equation a completely analytic solution for the drag coefficient of a disc in the infinite Mach number limit [20,21]. Therefore, by applying our method to a disc we shall be able to compare our method with that of Willis without any uncertainty associated with the finite precision of any numerical quadrature method and to extend his result to finite Mach numbers. The sphere drag was selected since we had previously studied the sphere drag in the zero and infinite Mach number limit [16,19] and since other investigators have studied the sphere drag by approximate methods [4,7,8,14,15].

CHAPTER II

DERIVATION OF COLLISION INTEGRALS FOR THE DRAG FORCE IN THE NEARLY FREE MOLECULAR FLOW REGIME†

2.1 Introduction

A theory of the force on an object in a gas stream in analogy with current procedures in the kinetic theory of moderately dense gases was proposed by Dorfman and Sengers [22] and formulated by McClure and Dorfman [17]. It is the purpose of this chapter to review the theory and to derive the general form of the collision integrals for the first inverse Knudsen number correction to the drag force. These formulas will then be analyzed and evaluated as a function of the speed ratio in the subsequent chapters.

Let us consider a system of N gas molecules and a macroscopic object in a volume Ω. The object is at rest and located at the origin of the coordinate system. We shall consider the limit in which the ratio of the mass of the object and the mass of the molecules becomes infinite. The momentum and position vectors of molecule i are indicated by $\vec{p}_i$ and $\vec{r}_i$, respectively. The total momentum of the gas at time t is given by

$$\vec{P}(t) = \sum_{i=1}^{N} \vec{p}_i(t) \quad . \tag{2-1}$$

† This chapter was prepared in collaboration with Professor J. R. Dorfman, Dr. C. F. McClure and Dr. W. A. Kuperman [17,19].

In order to evaluate the total momentum we use the methods of non-equilibrium statistical mechanics with the state of the gas being characterized by an N-particle distribution function $D_N(\Gamma_N;X;t)$. Here $\Gamma_N \equiv (x_1,\ldots x_N) \equiv (\vec{p}_1,\vec{r}_1,\ldots \vec{p}_N,\vec{r}_N)$ represents the 6N-dimensional phase space of the gas, X denotes the macroscopic object and $D_N(\Gamma_N;X;t)$ is defined such that $D_N(\Gamma_N;X;t)d\Gamma_N$ is the probability that the phases of the particles in the presence of the object will lie between Γ_N and $\Gamma_N + d\Gamma_N$ at time t. The distribution function is normalized such that

$$\int d\Gamma_N \, D_N(\Gamma_N;X;t) = 1. \tag{2-2}$$

The expectation value $\langle \vec{P}(t) \rangle$ is then determined by

$$\langle \vec{P}(t) \rangle = \int d\Gamma_N \sum_{i=1}^{N} \vec{p}_i D_N(\Gamma_N;X;t) \ . \tag{2-3}$$

Neglecting wall effects that disappear in the thermodynamic limit ($N\to\infty$, $\Omega\to\infty$, $N/\Omega=n$), the force $\vec{E}(t)$ on the object is equal to the negative of the time rate of change of this total momentum [17]

$$\vec{E}(t) = -\frac{d}{dt} \langle \vec{P}(t) \rangle \ . \tag{2-4}$$

The time evolution of the N-particle distribution function $D_N(\Gamma_N;t)$ in the *absence* of a foreign object is governed by the Liouville equation

$$\frac{\partial}{\partial t} D_N(\Gamma_N;t) + L_N(x_1\ldots x_N) \, D_N(\Gamma_N;t) = 0 \tag{2-5}$$

where L_N is the Liouville operator

$$L_N(x_1 \ldots x_N) \equiv \sum_{i=1}^{N} \frac{\vec{p}_i}{m} \cdot \frac{\partial}{\partial \vec{r}_i} - \sum_{i<j}^{N}\sum \theta_{ij} \tag{2-6}$$

Here θ_{ij} is a differential operator

$$\theta_{ij} \equiv \frac{\partial \phi_{ij}}{\partial \vec{r}_i} \cdot \frac{\partial}{\partial \vec{p}_i} + \frac{\partial \phi_{ij}}{\partial \vec{r}_j} \cdot \frac{\partial}{\partial \vec{p}_j} , \tag{2-7}$$

where $\phi_{ij} \doteq \phi(r_{ij}) = \phi(|\vec{r}_i - \vec{r}_j|)$ is the intermolecular potential between molecules i and j. The solution of the Liouville equation (2-5) may be written formally as

$$D_N(\Gamma_N;t) = e^{-tL_N(\Gamma_N)} D_N(\Gamma_N;0) \quad . \tag{2-8}$$

In the past decade methods have been developed for the evaluation of the average value of a phase function $A(x_1 \ldots x_N)$

$$\langle A \rangle = \int d\Gamma_N \, A(\Gamma_N) D_N(\Gamma_N;t) \quad . \tag{2-9}$$

In particular, Cohen and Green have formulated a cluster expansion of the streaming operator $\exp(-tL_N)$ that leads to a density expansion for $\langle A \rangle$ [23,24]. In order to apply the same method to the drag problem we need to extend the concept of streaming operator or time-displacement operator to a system of molecules in the presence of a foreign object.

2.2 Time Evolution of a System of Molecules in the Presence of an Object.

A description of the time evolution of a system of molecules in the presence of a foreign object must take into account the fact that the interaction of the molecules and the surface of the object is usually of a stochastic nature. A suitable time-displacement operator for such a

system was formulated by McClure and Dorfman [17].

Let $\Pi_{t'-t}(\Gamma'_N;t'|\Gamma_N;t)$ be the transition probability density that takes the system from the phase Γ_N at time t to the phase Γ'_N at time t'. The average total momentum of the system is given by

$$\langle\vec{P}(t)\rangle = \int d\Gamma_N \, \vec{P}(\Gamma_N) \, D_N(\Gamma_N;X;t), \tag{2-10}$$

where $\vec{P}(\Gamma_N) = \sum_{i=1}^{N} \vec{p}_i$. In terms of the transition probability Π_t we may rewrite (2-10) as

$$\langle\vec{P}(t)\rangle = \int d\Gamma_N \, \vec{P}(\Gamma_N) \int d\Gamma_N^{\circ} \, \Pi_t(\Gamma_N;t|\Gamma_N^{\circ};0) D_N(\Gamma_N^{\circ};X;0) \quad . \tag{2-11}$$

Introducing a time displacement operator as

$$\mathcal{T}_t \, \vec{P}(\Gamma_N^{\circ}) \equiv \int d\Gamma_N \vec{P}(\Gamma_N) \Pi_t(\Gamma_N;t|\Gamma_N^{\circ};0) \quad , \tag{2-12}$$

(2-11) becomes

$$\langle\vec{P}(t)\rangle = \int d\Gamma_N^{\circ} \, D_N(\Gamma_N^{\circ};X;0) \, \mathcal{T}_t \, \vec{P}(\Gamma_N^{0}) \; . \tag{2-13}$$

In order to formulate the time-displacement operator $\mathcal{T}_t$ we need to specify the interaction between the molecules and the surface of the object. For this purpose we assume that a gas molecule after colliding with the object with an incoming velocity $\vec{v}$, will be re-emitted from the surface with a velocity $\vec{v}'$ with a probability $\eta(\vec{v}'|\vec{v})d\vec{v}'$. We assume that all molecules impinging on the object will be re-emitted again, so that the transition probability is normalized as

$$\int d\vec{v}' \; \eta(\vec{v}'|\vec{v}) = 1 \tag{2-14}$$

To account for the interaction of a molecule, labeled 1, with the surface of the sphere it is convenient to introduce an operator T(1X), defined as [17,19]

$$T(1X) \equiv \int d\vec{v}' \int_{\vec{v}\cdot\hat{n}<0} d\vec{A} \; \eta(\vec{v}'|\vec{v}) \; |\vec{v}\cdot\hat{n}| \; \delta^3(\vec{r}-\vec{R}) \, (R_R - 1) \quad , \tag{2-15}$$

Here it is assumed that the molecule with incoming velocity $\vec{v}$ strikes the surface of the object at the position $\vec{R}$ inside the two-dimensional surface element $d\vec{A}$. The symbol $\delta^3(\vec{r}-\vec{R})$ represents a three-dimensional delta function. The operator $\mathcal{R}_R$ is a rotation operator that transforms the velocity $\vec{v}$ *before* the collision into the velocity $\vec{v}'$ *after* the collision of the molecule with the object. The vector $\hat{n}$ is a unit vector in the direction of the outward normal and the condition $\vec{v}\cdot\hat{n}<o$ indicates that the molecule strikes the surface from the outside. The integrations extend over all velocities $\vec{v}'$ and over the entire surface of the object.

We shall refer to T(1X), defined in (2-15), as the binary collision operator for a molecule and the object. It may be considered as the sum of two operators

$$T(1X) = T^i(1X) + T^n(1X) \quad , \tag{2-16}$$

where

$$T^i(1X) \equiv \int d\vec{v}' \int_{\vec{v}\cdot\hat{n}<o} d\vec{A}\ \eta(\vec{v}'|\vec{v})\,|\vec{v}\cdot\hat{n}|\,\delta^3(\vec{r}-\vec{R})\,\mathcal{R}_R \quad , \tag{2-16a}$$

$$T^n(1X) \equiv -\int_{\vec{v}\cdot\hat{n}<o} d\vec{A}\,|\vec{v}\cdot\hat{n}|\,\delta^3(\vec{r}-\vec{R}) \quad . \tag{2-16b}$$

The meaning of the operators $T^i(1X)$ and $T^n(1X)$ is illustrated in Fig. 1. In this figure the circle indicates the surface of the object and the lines represent the trajectory of the molecule. The operators $T^i(1X)$ and $T^n(1X)$ are only different from zero, if the molecule impinges on the object. The operator $T^i(1X)$ transforms the incident velocity $\vec{v}_1=\vec{v}$ of the molecule into the velocity $\vec{v}'$ of the molecule after being re-emitted from the object, taking into account the appropriate probability distribution for $\vec{v}'$. We

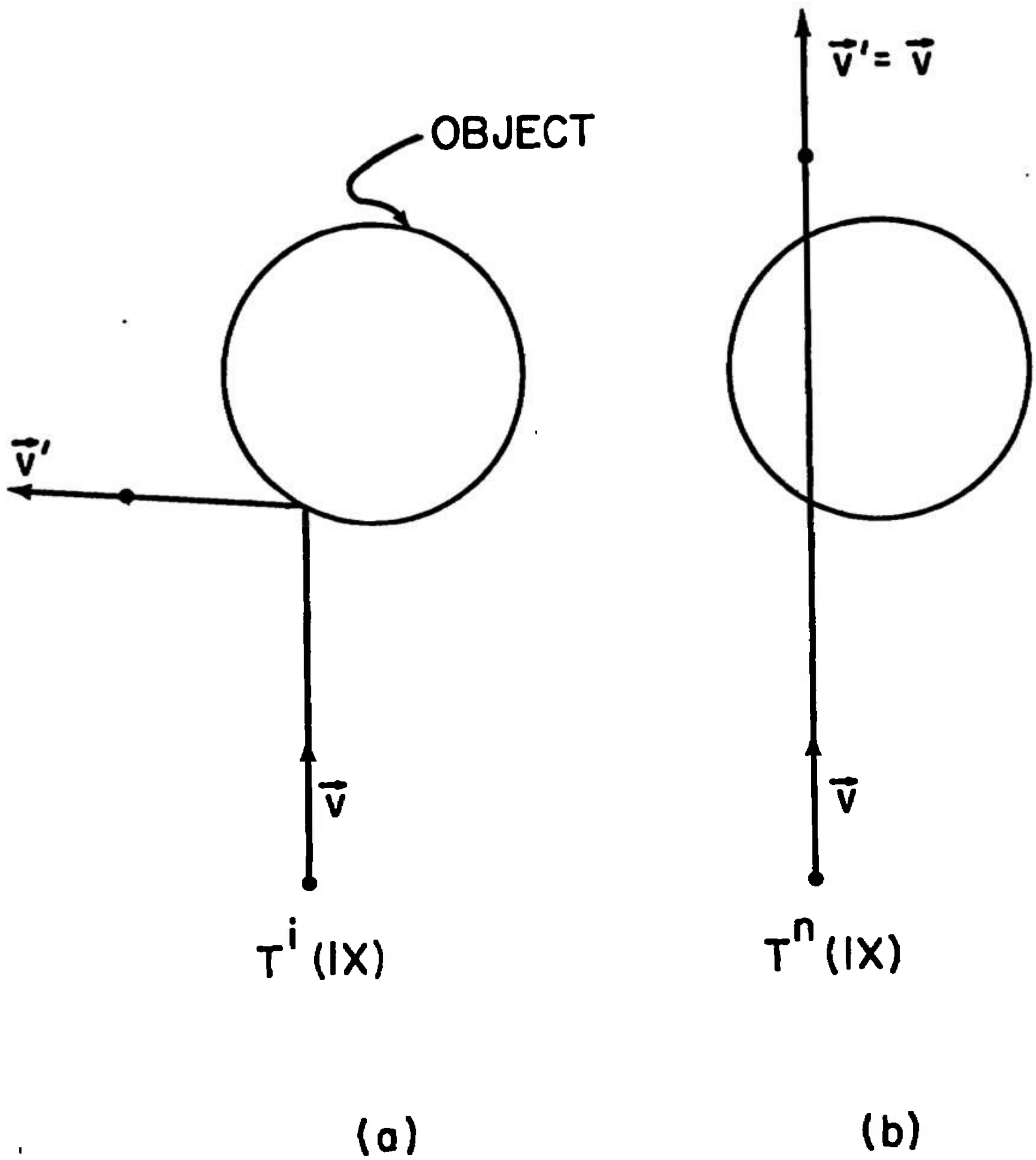


Figure 1. Schematic representation of the trajectory of a molecule in the presence of the object.
(a). An "interacting" collision with the object.
(b). A "non-interacting" collision with the object.

refer to this process as an "interacting" collision between the molecule and the object as shown in Fig. 1a. The operator $T^n(1X)$ requires also that the molecule collides with the object, but it does not change the velocity $\vec{v}$. The operator $T^n(1X)$, therefore, corresponds to a "non-interacting" collision in which the velocity of the molecular after the collision is equal to the velocity prior to the collision, as shown in Fig. 1b.

The operator $T(1X)$ is a generalization of the binary collision operators $T(ij)$ introduced by Ernst et al. to describe the interaction between two hard sphere molecules [25]. These binary collision operators $T(ij)$ were defined as

$$T(ij) = T^i(ij) + T^n(ij) \quad , \tag{2-17}$$

where $T^i(ij)$ corresponds to an interacting collision between two hard sphere molecules i and j

$$T^i(ij) = \sigma^2 \int_{\vec{v}_{ij}\cdot\hat{\sigma}_{ij}<0} d\hat{\sigma}_{ij} |\vec{v}_{ij}\cdot\hat{\sigma}_{ij}| \delta^3(\vec{r}_{ij}-\vec{\sigma}_{ij}) R_{\sigma_{ij}} \quad , \tag{2-17a}$$

and $T^n(ij)$ corresponds to a non-interacting collision between two hard sphere molecules

$$T^n(ij) = -\sigma^2 \int_{\vec{v}_{ij}\cdot\hat{\sigma}_{ij}<0} d\hat{\sigma}_{ij} |\vec{v}_{ij}\cdot\hat{\sigma}_{ij}| \delta^3(\vec{r}_{ij}-\hat{\sigma}_{ij}) \quad . \tag{2-17b}$$

Here $\vec{v}_{ij}=\vec{v}_i-\vec{v}_j$, $\vec{r}_{ij}=\vec{r}_i-\vec{r}_j$ are the relative velocity and position of the two molecules, σ is the diameter of the molecules and $\vec{\sigma}_{ij}$ is the perihelion vector of the collision between molecules i and j (= vector from center of j to center of i at time of contact). The rotation operator $R_{\sigma_{ij}}$ transforms the velocities $\vec{v}_i$, $\vec{v}_j$ *prior* to the collision into the velocities $\vec{v}_i'$, $\vec{v}_j'$ *after* the collision†

---

† In this report a symbol $\hat{a}$ always indicates the unit vector in the direction of the vector $\vec{a}$.

$$\vec{v}_i' = R_{\sigma_{ij}} \vec{v}_i = \vec{v}_i - (\vec{v}_{ij} \cdot \hat{\sigma}_{ij})\hat{\sigma}_{ij} \; ,$$
$$\vec{v}_j' = R_{\sigma_{ij}} \vec{v}_j = \vec{v}_j + (\vec{v}_{ij} \cdot \hat{\sigma}_{ij})\hat{\sigma}_{ij} \; . \tag{2-18}$$

The properties of these binary collision operators were discussed in detail in previous reports [26,27].

As shown by McClure and Dorfman [17] and also discussed by Kuperman [19] the time-displacement operator, defined in (2-12), can be expressed in terms of these binary collision operators as

$$\mathcal{T}_t(1,\ldots,N;X) = e^{t[G_N + \sum_{k=1}^{N} T(kX)]} \quad , \tag{2-19}$$

where G_N is the same resolvent operator as used in AEDC-TR-72-142 [27]

$$G_N(1,\ldots N) = \sum_{i=1}^{N} \frac{\vec{p}_i}{m} \cdot \frac{\partial}{\partial \vec{r}_i} + \sum_{i<j}\sum^{N} T(ij) \quad . \tag{2-20}$$

In the absence of the object the time-displacement operator reduces to

$$\mathcal{T}_t(1,\ldots N) = e^{tG_N} \quad . \tag{2-21}$$

The explicit expressions (2-17) for the binary collision operators T(ij) refer to a gas of hard spheres. However, the crucial assumption enabling us to represent the time-displacement operators by (2-19) and (2-20) is the assumption that the size of the molecules is small compared to the mean free path. Since we shall apply the theory to the case that even the macroscopic object is small compared to the mean free path, this assumption is satisfied for any gas of molecules with finite interaction range. Therefore, we may use (2-19) and (2-20) to represent the time-displacement operator for any gas of molecules in the nearly free molecular flow regime, provided that the hard sphere cross section in (2-17) is replaced with the actual cross section of the molecules under consideration. For the same reason we do not

distinguish between T(ij) and $\bar{T}$(ij) operators as was needed in the theory of transport properties of moderately dense gases [27].

2.3 Cluster Expansion.

According to (2-4) and (2-13) the force $\vec{E}(t)$ on the object is

$$\vec{E}(t) = -\frac{d}{dt}\int d\Gamma_N \, D_N(\Gamma_N;X;0) \quad \mathcal{T}_t(1,\ldots N;X) \; \vec{P}(\Gamma_N) \quad . \tag{2-22}$$

Using the form (2-19) for the time-displacement operator $\mathcal{T}_t$ we thus obtain

$$\vec{E}(t) = -\int d\Gamma_N \, D_N(\Gamma_N;X;0) \; \mathcal{T}_t\left[G_N + \sum_{k=1}^{N} T(kX)\right]\vec{P} \quad . \tag{2-23}$$

The operator G_N is associated with the streaming of the molecules without any interactions with the object. In the absence of the object the total momentum $\vec{P}$ is conserved, so that G_N and $\vec{P}$ commute: $G_N\vec{P} = \vec{P}G_N = 0$. Moreover $T(kX)\vec{p}_\ell = 0$ for $\ell \neq k$ and (2-23) becomes

$$\vec{E}(t) = -\sum_{k=1}^{N}\int d\Gamma_N \, D_N(\Gamma_N;X;0) \; \mathcal{T}_t(1,\ldots,N;X)T(kX)\vec{p}_k \quad , \tag{2-24}$$

which for a gas of N identical molecules reduces to

$$\vec{E}(t) = -N\int d\Gamma_N \, D_N(\Gamma_N;X;0) \; \mathcal{T}_t(1,\ldots,N;X)T(1X)\vec{p}_1 \quad . \tag{2-25}$$

We now write the time-displacement operator $\mathcal{T}_t(1,\ldots N;X)$ in the form of a cluster expansion [17,19].

$$\begin{aligned}\mathcal{T}_t(1,\ldots,N;X) &= U_t(1)\mathcal{T}_t(2,\ldots,N;X) + U_t(1;X)\mathcal{T}_t(2,\ldots,N) + \\ &+ \sum_{i=2}^{N}\Big[U_t(1i)\mathcal{T}_t(2,\ldots,i-1,i+1,\ldots,N;X) + U_t(1i;X)\mathcal{T}_t(2,\ldots,i-1,i+1,\ldots,N)\Big] \\ &+ \ldots + U_t(1,\ldots,N;X) \quad ,\end{aligned} \tag{2-26}$$

Here we are introducing Ursell operators defined by

$$\begin{aligned} U_t(1) &= \mathcal{T}_t(1) \quad , \\ U_t(1;X) &= \mathcal{T}_t(1;X) - \mathcal{T}_t(1) \quad , \\ U_t(12) &= \mathcal{T}_t(12) - \mathcal{T}_t(1)\mathcal{T}_t(2) \quad , \\ U_t(12;X) &= \mathcal{T}_t(12;X) - \mathcal{T}_t(1;X)\mathcal{T}_t(2) - \mathcal{T}_t(2;X)\mathcal{T}_t(1) \\ &\qquad - \mathcal{T}_t(12) + 2\,\mathcal{T}_t(1)\mathcal{T}_t(2) \quad , \\ &\ldots, \text{etc.} \end{aligned} \tag{2-27}$$

We substitute the expansion (2-26) for the time-displacement operator $\mathcal{T}_t(1,\ldots N;X)$ into the expression (2-25) for the drag force. Since the $\mathcal{T}_t$ operators in (2-26) do not affect the momentum $\vec{p}_1$ of molecule 1, we obtain

$$\vec{E}(t) = -N\int d\Gamma_N\, D_N(\Gamma_N;X;0)\Big[\{U_t(1) + U_t(1;X)\} + (N-1)\ \{U_t(12) + U_t(12;X)\} + \ldots + U_t(1,\ldots N;X)\Big]T(1X)\vec{p}_1\ , \qquad (2\text{-}28)$$

where we have again used the fact that the gas consists of N identical molecules. Introducing reduced distribution functions defined as

$$F_s(x_1,\ldots,x_s;X;t) = \Omega^s\int dx_{s+1}\ldots dx_N\, D_N(\Gamma_N;X;t)\quad , \qquad (2\text{-}29)$$

we obtain in the thermodynamic limit $N\to\infty$, $\Omega\to\infty$, $N/\Omega=n$

$$\begin{aligned}\vec{E}(t) = &-n\int dx_1\, F_1(x_1;X;0)\ [U_t(1)+U_t(1;X)]T(1X)\vec{p}_1 + \\ &-n^2\int dx_1 dx_2\, F_2(x_1,x_2;X;0)\,[U_t(12)+ U_t(12;X)]\ T(1X)\vec{p}_1 + \\ &+ \ldots\ .\end{aligned} \qquad (2\text{-}30)$$

This expression relates the force to the distribution functions $F_s(x_1\ldots x_s;X;0)$ at the initial state. Following McClure and Dorfman we assume that the initial state is unaffected by the presence of the object

$$F_s(x_1\ldots x_s;X;0) = F_s(x_1\ldots x_s;0)\quad . \qquad (2\text{-}31)$$

We also introduce the usual assumption that the molecules are uncorrelated initially, so that $F_s(x_1\ldots x_s;0)$ may be approximated by a product of single-particle distribution functions

$$F_s(x_1\ldots x_s;0) = \prod_{i=1}^{s} F_1(x_i;0)\quad . \qquad (2\text{-}32)$$

The extent to which this assumption is justified has been discussed by Dorfman and Cohen [28]. In the absence of the foreign object the time-displacement operator $\mathcal{T}_t(1,\ldots N)$ may be reversed in time; therefore the single-particle distribution function $F_1(x_1;t)$ in the absence of

the object may be written

$$F_1(x_1;t) = \Omega\int dx_2 \ldots dx_N \, \mathcal{T}_{-t}(1,\ldots N) \; D_N(\Gamma_N;X;0) \quad , \tag{2-33}$$

or with (2-26) and (2-29)

$$F_1(x_1;t) = U_{-t}(1)\; F_1(x_1;0) + n\int dx_2 U_{-t}(12) F_2(x_1,x_2;0) \quad + \; . \; . \tag{2-34}$$

Using (2-32) we can invert (2-34) to yield an expansion for $F_1(x_1;0)^\dagger$

$$F_1(x_1;0) = U_{+t}(1)F_1(x_1;t) - nU_{+t}(1)\int dx_2 U_{-t}(12) \prod_{i=1}^{2} U_{+t}(i)F_1(x_i;t) + \\ + \; . \; . \; . \tag{2-35}$$

Substitution of (2-31), (2-32) and (2-35) into (2-30) yields a density expansion for the force $\vec{E}(t)$.

In this report we consider the drag force $\vec{E}$ in the steady state which corresponds to the limit $t\to\infty$. In this limit the gas is in equilibrium in the absence of the object and the single-particle distribution function $F_1(x_i;t)$ becomes independent of the position $\vec{r}_i$ and time t and may be represented by a displaced Maxwell distribution

$$\lim_{t\to\infty} F_1(x_i;t) = F(\vec{v}_i;\vec{V}) = (2\pi mkT)^{-3/2} e^{-\frac{m(\vec{v}_i-\vec{V})^2}{2kT}} \quad . \tag{2-36}$$

In this limit we obtain for the density expansion of the force $\vec{E}$ on the object

$$\vec{E} = \lim_{t\to\infty} \sum_\alpha \vec{E}_\alpha(t), \tag{2-37}$$

where, for the purposes of this report, we consider only the first two terms $\vec{E}_o$ and $\vec{E}_1$.

$$\vec{E}_o(t) = -n\int dx_1 \; F(\vec{v}_1;\vec{V}) \{U_t(1) + U_t(1;X)\} \mathcal{T}(1X)\vec{p}_1 \quad , \tag{2-38a}$$

$$\vec{E}_1(t) = -n^2\int dx_1 dx_2 \; F(\vec{v}_1;\vec{V})F(\vec{v}_2;\vec{V}) \{U_t(12) + U_t(12;X)\} \mathcal{T}(1X)\vec{p}_1 + \\ + n^2\int dx_1 dx_2 [U_t(1)U_{-t}(12)F(\vec{v}_1;\vec{V})F(\vec{v}_2;\vec{V})] \{U_t(1) + U_t(1;X)\} \mathcal{T}(1X)\vec{p}_1 . \tag{2-38b}$$

† The operators $U_t(1\ldots s)$ do not involve the object and the corresponding time reversed operators $U_{-t}(1\ldots s)$ are well defined. The operators $U_t(1\ldots s;X)$ on the other hand depend on the stochastic interaction of the molecules with the object and therefore cannot be reversed in time [17].

Here we use the convention that operators inside the square brackets [ ] do not operate on terms outside these brackets. In deriving (2-38) use is made of the fact that $U_t(i)F(\vec{v}_i;\vec{V}) = F(\vec{v}_i;\vec{V})$. Since the adjoint operator $(U_{+t}(1)U_{-t}(12))^{\dagger} = U_{+t}(12)U_{-t}(1)$ and since $U_{-t}(1)U_{+t}(1) = 1$, (2-38b) reduces to

$$\vec{E}_1(t) = -n^2\int dx_1 dx_2 \; F(\vec{v}_1;\vec{V})F(\vec{v}_2;\vec{V})\{U_t(12;X)-U_t(12)U_{-t}(1)U_t(1;X)\}\mathcal{T}(1X)\vec{p}_1 . \tag{2-38c}$$

2.4 Binary Collision Expansion

The various terms $\vec{E}_\alpha(t)$ in the expansion (2-37) for the force $\vec{E}$ contain the dynamics of α+1 molecules and the object. When the density of the gas is sufficiently small so that the mean free path is large compared to the size of the molecules we may neglect the probability that two collisions occur simultaneously and the dynamical processes reduce to sequences of successive collisions among the molecules and the object. In order to classify the various collision sequences that contribute to the drag it is convenient to represent the time-displacement operators $\mathcal{T}_t(1...s)$ and $\mathcal{T}_t(1...s;X)$ in terms of a binary collision expansion.

A precise formulation of the binary collision expansion for the time-displacement operator of a gas of hard spheres in the absence of an object was developed by Ernst et al. [25]. The extension of this procedure to the time-displacement operator in the presence of the object was formulated by McClure and Dorfman [17] and further discussed by Kuperman [19]. Following these authors we note that the time-displacement operator (2-19) has the form

$$\mathcal{T}_t(1\ldots s) = e^{t[L^\circ(1\ldots s)+K(1\ldots s;X)]}, \tag{2-39}$$

where

$$L^\circ(1\ldots s) = \sum_{i=1}^{s} \frac{\vec{p}_i}{m} \cdot \frac{\partial}{\partial \vec{r}_i} \tag{2-40}$$

and

$$K(1\ldots s;X) = \sum_{i<j}^{s}\sum T(ij) + \sum_{k=1}^{s} T(kX) . \tag{2-41}$$

The time-displacement operator satisfies the relation

$$\mathcal{T}_t = S^\circ_{+t} + \int_0^t d\tau \; S^\circ_{t-\tau} \; K \; \mathcal{T}_\tau , \tag{2-42}$$

where S°_t is the streaming operator representing the free streaming of the molecules without any interactions

$$S^\circ_t(1\ldots s) = e^{-tL^\circ(1\ldots s)}. \tag{2-43}$$

The binary collision expansion is obtained after successive iteration of (2-42)

$$\mathcal{T}_t = S^\circ_t + \int_0^t d\tau_1 \; S^\circ_{t-\tau_1} K \; S^\circ_{\tau_1} + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \; S^\circ_{t-\tau_1} K \; S^\circ_{\tau_1-\tau_2} K \; S^\circ_{\tau_2} + \ldots \tag{2-44}$$

Introducing the convolution product

$$f * g = \int_0^t d\tau \; f(\tau) g(t-\tau) = \int_0^t d\tau' \; f(t-\tau') g(\tau') , \tag{2-45}$$

where $f(t)$ and $g(t)$ are functions of the time t, the binary collision expansion (2-44) may be written as

$$\mathcal{T}_t = S^\circ_t + S^\circ * KS^\circ + S^\circ * KS^\circ * KS^\circ + \ldots \tag{2-46}$$

The operator K represents a collision either between two molecules or a molecule and the object and the operator S°_t represents the free streaming of the molecules between collisions. Thus the terms in the binary collision expansion (2-46) correspond to sequences of zero, one, two, etc. successive collisions among the s molecules and the object.

However, only those terms in (2-46) will be different from zero which correspond to sequences of collisions that are permitted by the laws of mechanics. The number of physically possible sequences of collisions depends on the geometry of the object. In this report we shall limit our considerations to convex objects. A molecule emitted by a convex object cannot return to the surface of the object unless it first collides with another molecule, so that

$$S^{\circ *}T(iX)S^{\circ *}T(iX)S^{\circ} = 0. \tag{2-47}$$

Furthermore, the expansion parameter is the inverse Knudsen number $n\sigma^2 L$, where L measures the size of the object. Terms in (2-46) that contain more than one binary collision operator corresponding to a collision between two molecules may be neglected since they are of higher order in $n\sigma^3$. A complete list of the combinations of binary collision operators that yield vanishing contributions was presented by Kuperman [19].

Substitution of (2-27) into (2-38) yields

$$\vec{E}_o(t) = -n\int dx_1\ F(\vec{v}_1;\vec{V})\ \mathcal{T}_t(1;X)\vec{p}_1\ , \tag{2-48a}$$

$$\vec{E}_1(t) = -n^2\int dx_1 dx_2\ F(\vec{v}_1;\vec{V})F(\vec{v}_2;\vec{V}) \tag{2-48b}$$

$$\cdot\{\mathcal{T}_t(12;X)-\mathcal{T}_t(1)\mathcal{T}_t(2;X)-\mathcal{T}_t(1;X)\mathcal{T}_t(2)-\mathcal{T}_t(12)+2\mathcal{T}_t(1)\mathcal{T}_t(2)\}T(1X)\vec{p}_1 .$$

Introduction of the binary collision expansion (2-46) into (2-48) then leads to [17,19]

$$\vec{E}_o(t) = -n\int dx_1\ F(\vec{v}_1;\vec{V})T(1X)\vec{p}_1\ , \tag{2-49a}$$

$$\vec{E}_1(t) = -n^2\int dx_1 dx_2\ F(\vec{v}_1;\vec{V})F(\vec{v}_2;\vec{V})\ \ (\vec{B}_3+\vec{B}_4)\ , \tag{2-49b}$$

with

$$\vec{B}_3 = S^{\circ *}K(12;X)S^{\circ *}K(12;X)S^{\circ}T(1X)\vec{p}_1\ , \tag{2-50a}$$

$$\vec{B}_4 = S^{\circ *}K(12;X)S^{\circ *}K(12;X)S^{\circ *}K(12;X)S^{\circ}T(1X)\vec{p}_1\ . \tag{2-50b}$$

For convex objects the binary collision expansion of $\vec{E}_o$ terminates after the first term at a result of (2-47) and the binary collision expansion of $\vec{E}_1$ terminates after the second term as shown by Kuperman [19]. If we substitute $K(12;X) = T(12)+T(1X)+T(2X)$ into (2-50) and omit all vanishing terms, $\vec{B}_3$ and $\vec{B}_4$ reduce to

$$\vec{B}_3 = S^{\circ*}\{T(1X)+T(2X)\}S^{\circ*}T(12)S^{\circ}T(1X)\vec{p}_1 \ , \tag{2-51a}$$

$$\vec{B}_4 = S^{\circ*}\{T(1X)S^{\circ*}T(2X)+T(2X)S^{\circ*}T(1X)\}S^{\circ*}T(12)S^{\circ}T(1X)\vec{p}_1 . \tag{2-51b}$$

Since we may interchange the integration variables x_1 and x_2, we adopt the convention that molecule 1 is identified as the molecule in the left most T(iX) operator, i.e. molecule 1 is the molecule that collides initially with the object. We may therefore replace (2-51) with

$$\vec{B}_3 = S^{\circ*}T(1X)S^{\circ*}T(12)S^{\circ}\{T(1X)+T(2X)\}\{\vec{p}_1+\vec{p}_2\}, \tag{2-52a}$$

$$\vec{B}_4 = S^{\circ*}T(1X)S^{\circ*}T(2X)S^{\circ*}T(12)S^{\circ}\{T(1X)+T(2X)\}\{\vec{p}_1+\vec{p}_2\}. \tag{2-52b}$$

The term $\vec{B}_3$ accounts for the effect of sequences of *three* successive collisions among two molecules, molecules 1 and 2, and the object, and the term $\vec{B}_4$ is related to sequences of *four* successive collisions among two molecules and the object. A complete classification of these collision sequences was made by Kuperman [19]. In order to enumerate these collision sequences, we use (2-16) and (2-17) to express $\vec{B}_3$ and $\vec{B}_4$ more explicitly as

$$\begin{aligned}
\vec{B}_3 = \ & S^{\circ*}\, T^{i}(1X)S^{\circ*}T^{i}(12)S^{\circ}T(1X)\vec{p}_1 + && \text{(R1)}\\
& + S^{\circ*}\, T^{n}(1X)S^{\circ*}T^{i}(12)S^{\circ}T(1X)\vec{p}_1 + && \text{(R2)}\\
& + S^{\circ*}\, T^{i}(1X)S^{\circ*}T^{i}(12)S^{\circ}T(2X)\vec{p}_2 + && \text{(C1)}\\
& + S^{\circ*}\, T^{n}(1X)S^{\circ*}T^{i}(12)S^{\circ}T(2X)\vec{p}_2 + && \text{(C2)}\\
& + S^{\circ*}\, T^{i}(1X)S^{\circ*}T^{n}(12)S^{\circ}T(2X)\vec{p}_2 + && \text{(H1)}\\
& + S^{\circ*}\, T^{n}(1X)S^{\circ*}T^{n}(12)S^{\circ}T(2X)\vec{p}_2 \ , && \text{(H2)}
\end{aligned} \tag{2-53}$$

and

$$
\begin{aligned}
\vec{B}_4 = \; & S^\circ * T^i(1X) S^\circ * T^i(2X) S^\circ * T^i(12) S^\circ T(1X) \vec{p}_1 + \quad (R1C1) \\
+ \; & S^\circ * T^n(1X) S^\circ * T^i(2X) S^\circ * T^i(12) S^\circ T(1X) \vec{p}_1 + \quad (R2C1) \\
+ \; & S^\circ * T^i(1X) S^\circ * T^n(2X) S^\circ * T^i(12) S^\circ T(1X) \vec{p}_1 + \quad (R1C2) \\
+ \; & S^\circ * T^n(1X) S^\circ * T^n(2X) S^\circ * T^i(12) S^\circ T(1X) \vec{p}_1 + \quad (R2C2) \\
+ \; & S^\circ * T^i(1X) S^\circ * T^i(2X) S^\circ * T^i(12) S^\circ T(2X) \vec{p}_2 + \quad (C1R1) \\
+ \; & S^\circ * T^n(1X) S^\circ * T^i(2X) S^\circ * T^i(12) S^\circ T(2X) \vec{p}_2 + \quad (C2R1) \\
+ \; & S^\circ * T^i(1X) S^\circ * T^n(2X) S^\circ * T^i(12) S^\circ T(2X) \vec{p}_2 + \quad (C1R2) \\
+ \; & S^\circ * T^n(1X) S^\circ * T^n(2X) S^\circ * T^i(12) S^\circ T(2X) \vec{p}_2 + \quad (C2R2) \; . \qquad (2\text{-}54)
\end{aligned}
$$

According to (2-53) $\vec{B}_3$ can be decomposed into a sum of six terms corresponding to six different types of sequences of three successive collisions among two molecules and the object. In analogy to the previous work of Sengers [29,30] we refer to these sequences as R1 and R2 (recollisions), C1 and C2 (cyclic collisions) and H1 and H2 (hypothetical collisions). These collision sequences are represented by the six diagrams in Fig. 2. In these diagrams the lines with labels 1 and 2 indicate the trajectories of molecules 1 and 2. The vertical line represents the object. The molecules traverse their trajectories in the direction indicated by arrows. An interacting collision between the two molecules or a molecule and the object causes a change in the direction of the trajectories. In a non-interacting collision the molecules continue to proceed in the direction of their original trajectories; in the diagrams we add a shaded region wherever we want to indicate the occurrence of a non-interacting collision.

Molecule 1 initially strikes the object and is either reflected by the object (R1, C1, H1) or passes through the object (R2, C2, H2). It then collides with molecule 2 such that molecule 1 either collides with

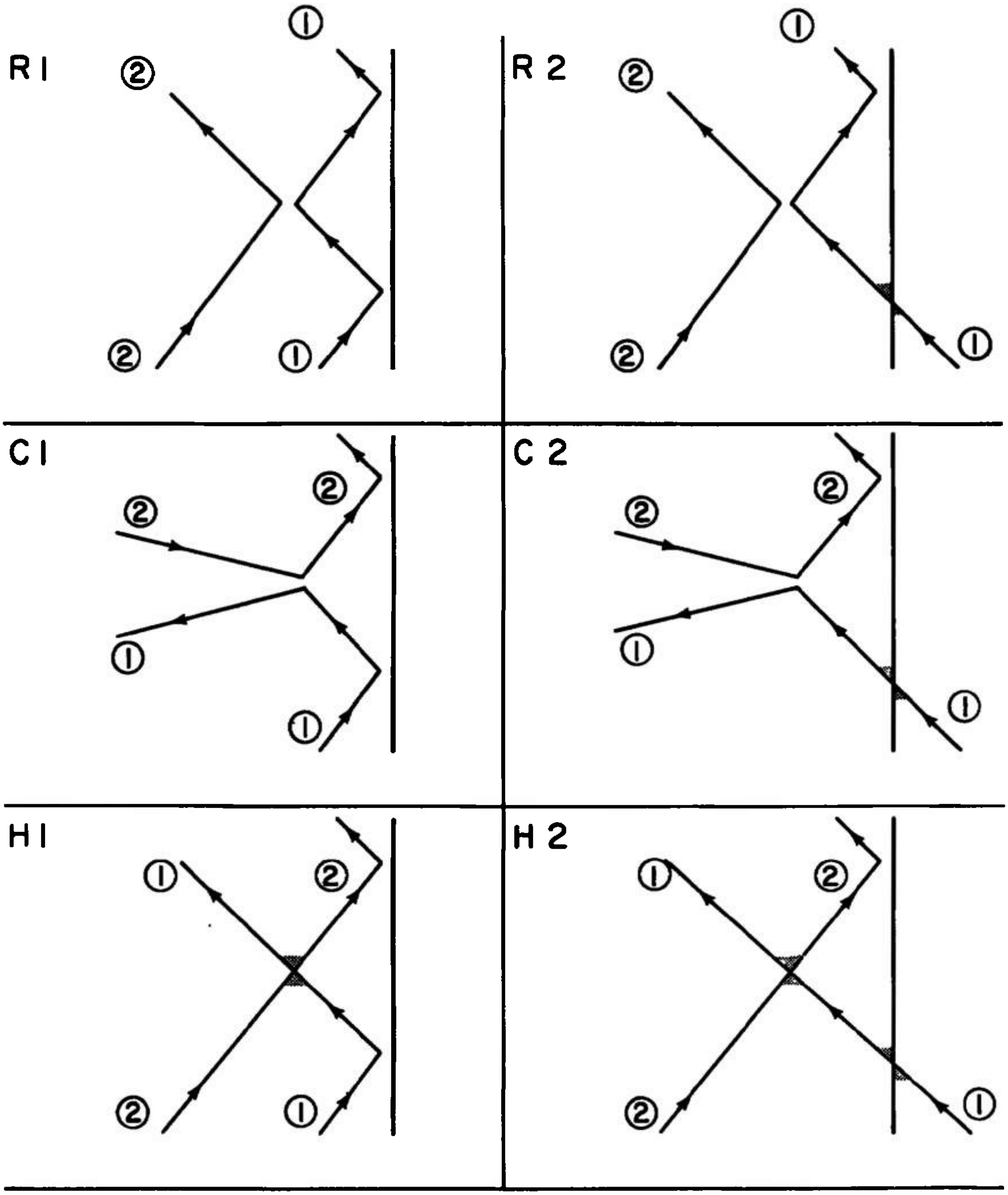


Figure 2. Sequences of three successive collisions among two molecules and the object.

the object again (R1, R2), or it causes molecule 2 to collide with the object (C1, C2), or it prevents molecule 2 from colliding with the object (H1, H2). On comparing Fig. 2 with Fig. 5 of AEDC-TR-69-68 [30] we note that these collision sequences are closely analogous to the collision sequences earlier encountered in the first density correction to the transport properties of a moderately dense gas. In fact, the new collision sequences are obtained if in the earlier three-particle collision sequences one of the molecules is replaced with the object.

According to (2-54) $\vec{B}_4$ can be decomposed into a sum of terms corresponding to eight different types of four successive collisions among two molecules and the object. These sequences are represented schematically by the diagrams of Fig. 3. As noted earlier, for convex objects we do not have to consider any sequences of five successive collisions among two molecules and the object in calculating the first inverse Knudsen number correction to the drag force.

The relative magnitude of $\vec{B}_3$ and $\vec{B}_4$ is determined by the probability that the corresponding collision sequences will occur. In evaluating the first density correction to the transport properties of a gas of hard spheres Gillespie and Sengers noted that the contribution from sequences of four successive collisions is only 10^{-4} times the contribution from sequences of three successive collisions [31]. A preliminary numerical analysis, conducted by Kuperman indicated that for the drag problem sequences of four successive collisions would yield a correction of less than one percent [19]. Therefore, to obtain a realistic estimate of the magnitude of the drag force we shall only consider the contributions from three successive collisions and approximate (2-49b) by

$$\vec{E}_1(t) \simeq -n^2 \int dx_1 dx_2 \; F(\vec{v}_1;\vec{V}) F(\vec{v}_2;\vec{V}) \vec{B}_3 \,. \qquad (2\text{-}55)$$

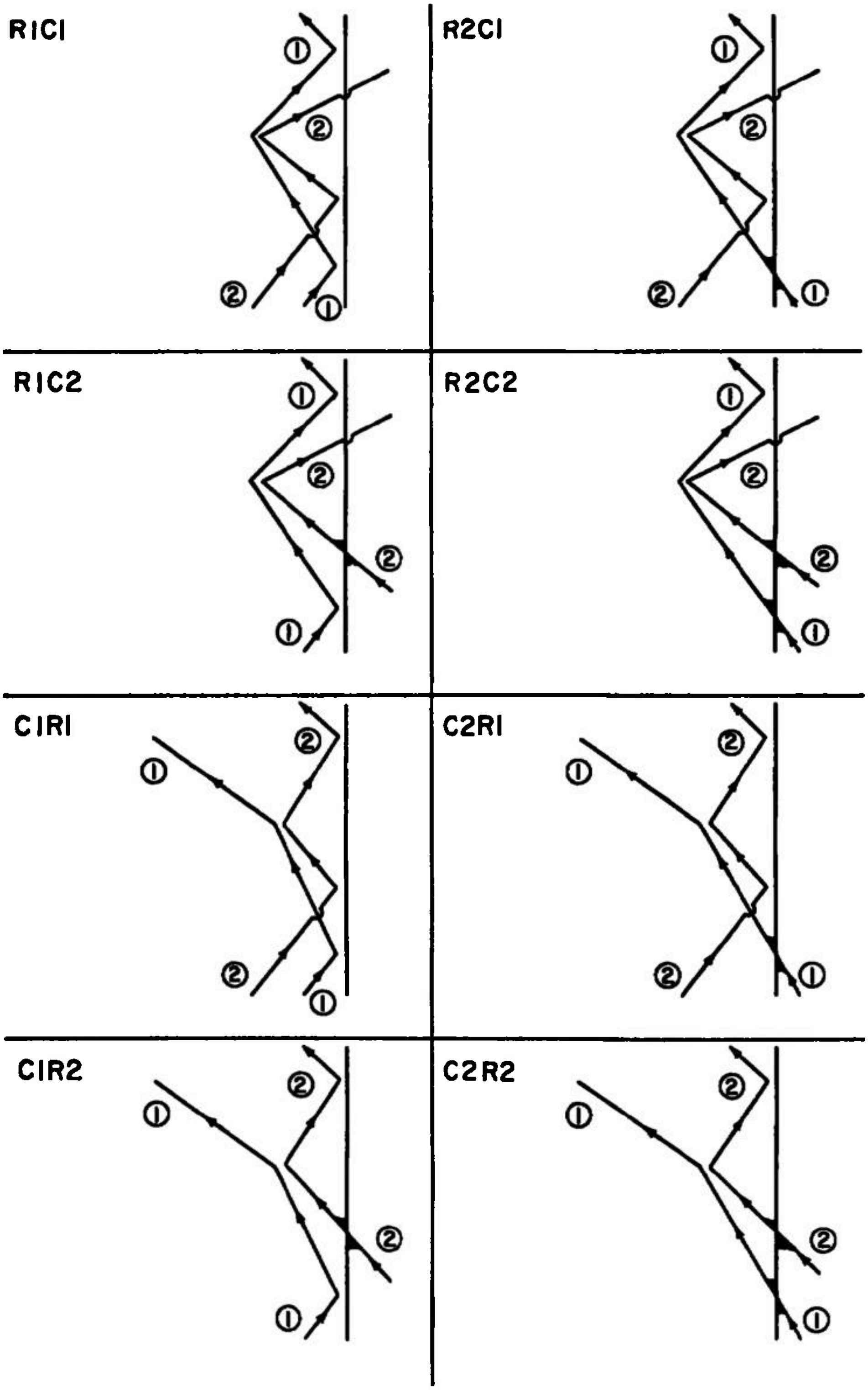


Figure 3. Sequences of four successive collisions among two molecules and the object.

2.5 Explicit Formulation of Collision Integrals

The drag force in the free molecular flow regime is determined by (2-49a). Introducing the explicit form (2-16) for the binary collision operator T(1X) and replacing the momentum integration with a velocity integration, we thus obtain

$$\vec{E}_o = -mn\int d\vec{v}_1 f(\vec{v}_1;\vec{V})\int_{\vec{v}_1\cdot\hat{n}_1<o} d\vec{A}_1\ |\vec{v}_1\cdot\hat{n}_1|\int d\vec{v}_1'\ \eta(\vec{v}_1'|\vec{v}_1)(\vec{v}_1'-\vec{v}_1)\ , \qquad (2\text{-}56)$$

with

$$f(\vec{v}_i;\vec{V}) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{m(\vec{v}_i-\vec{V})^2}{2kT}} \qquad (2\text{-}57)$$

The force $\vec{E}_1$, to be added in the nearly free molecular flow regime is given by (2-55) and can be decomposed into

$$\vec{E}_1 = \vec{E}_{R1} + \vec{E}_{R2} + \vec{E}_{C1} + \vec{E}_{C2} + \vec{E}_{H1} + \vec{E}_{H2}\ , \qquad (2\text{-}58)$$

where each term is uniquely related to the corresponding term in (2-53).

Each term in (2-58) may be written explicitly as a collision integral in terms of the initial velocities, intermediate velocities and final velocities in the collision sequences of Fig. 2 as shown by Kuperman [19]. As an example we consider the contribution $\vec{E}_{R1}$. Using the definition (2-45) of the convolution product, $\vec{E}_{R1}$ may be written as

$$\vec{E}_{R1} = -mn^2\lim_{t\to\infty}\int d\vec{r}_1 d\vec{r}_2 d\vec{v}_1 d\vec{v}_2\ f(\vec{v}_1;\vec{V})f(\vec{v}_2;\vec{V})$$
$$\cdot\int_0^t d\tau\int_0^\tau d\tau'\ S^\circ_{t-\tau}\ T^i(1X)\ S^\circ_{\tau-\tau'}\ T^i(12)\ S^\circ_{\tau'}\ T(1X)\vec{v}_1. \qquad (2\text{-}59)$$

It is convenient to introduce the transformation $\tau_1 = \tau-\tau'$ and $\tau_2 = \tau'$, so that

$$\vec{E}_{R1} = -mn^2\lim_{t\to\infty}\int d\vec{r}_1 d\vec{r}_2 d\vec{v}_1 d\vec{v}_2\ f(\vec{v}_1;\vec{V})f(\vec{v}_2;\vec{V})$$
$$\cdot\int_0^t d\tau_2\int_0^{t-\tau_2} d\tau_1\ S^\circ_{t-(\tau_1+\tau_2)}\ T^i(1X)\ S^\circ_{\tau_1}\ T^i(12)\ S^\circ_{\tau_2}\ T(1X)\vec{v}_1. \qquad (2\text{-}60)$$

The operators $S^\circ(12)$ are free streaming operators, so that

$$S^\circ_\tau \vec{r}_1 = \vec{r}_1 + \vec{v}_1\tau \quad , \qquad S^\circ_\tau \vec{v}_1 = \vec{v}_1 \; , \tag{2-61}$$
$$S^\circ_\tau \vec{r}_2 = \vec{r}_2 + \vec{v}_2\tau \quad , \qquad S^\circ_\tau \vec{v}_2 = \vec{v}_2 \; .$$

The left most operator $S^\circ_{t-(\tau_1+\tau_2)}$ in (2-60) translates the integrand according to the free particle motion (2-61). Since the integration extends over the entire phase space of molecules 1 and 2, the integral is invariant under such a transformation and (2-60) reduces to

$$\vec{E}_{R1} = -mn^2\int d\vec{r}_1 d\vec{r}_2 d\vec{v}_1 d\vec{v}_2 f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_2 \int_0^\infty d\tau_1 \; T^i(1X) S^\circ_{\tau_1} T^i(12) S^\circ_{\tau_2} T(1X)\vec{v}_1 , \tag{2-62}$$

In Fig. 4 we consider again a typical recollision. The lines represent the trajectories of molecules 1 and 2 and the circle indicates the object. Molecule 1 collides with the surface of the object at the position $\vec{R}_1$ (first collision), it subsequently collides with molecule 2 (second collision) and then it collides with the object again at the position $\vec{R}_2$ (third collision). The first collision may either be an interacting collision (Fig. 4a) or a non-interacting collision (Fig. 4b). We indicate the initial velocities of the molecules by $\vec{v}_1$, $\vec{v}_2$, the velocities after the first collision by $\vec{v}_1'$, $\vec{v}_2'$, the velocities after the second collision by $\vec{v}_1''$, $\vec{v}_2''$ and the velocities after the third collision by $\vec{v}_1'''$, $\vec{v}_2'''$; we indicate the relative velocities similarly by $\vec{v}_{12} = \vec{v}_1 - \vec{v}_2$, $\vec{v}_{12}' = \vec{v}_1' - \vec{v}_2'$, $\vec{v}_{12}'' = \vec{v}_1'' - \vec{v}_2''$, $\vec{v}_{12}''' = \vec{v}_1''' - \vec{v}_2'''$. We note that for the recollisions

$$\vec{v}_2' = \vec{v}_2 , \; \vec{v}_2''' = \vec{v}_2'' \qquad \text{(R1-sequence)} , \tag{2-63a}$$

and

$$\vec{v}_1' = \vec{v}_1 , \; \vec{v}_2' = \vec{v}_2 , \; \vec{v}_2''' = \vec{v}_2'' \qquad \text{(R2-sequence)} . \tag{2-63b}$$

Using this notation and the definitions (2-16), (2-17) and (2-61) of the operators $T(iX)$, $T(12)$ and $S^\circ(12)$, the expression for $\vec{E}_{R1}$ may be written

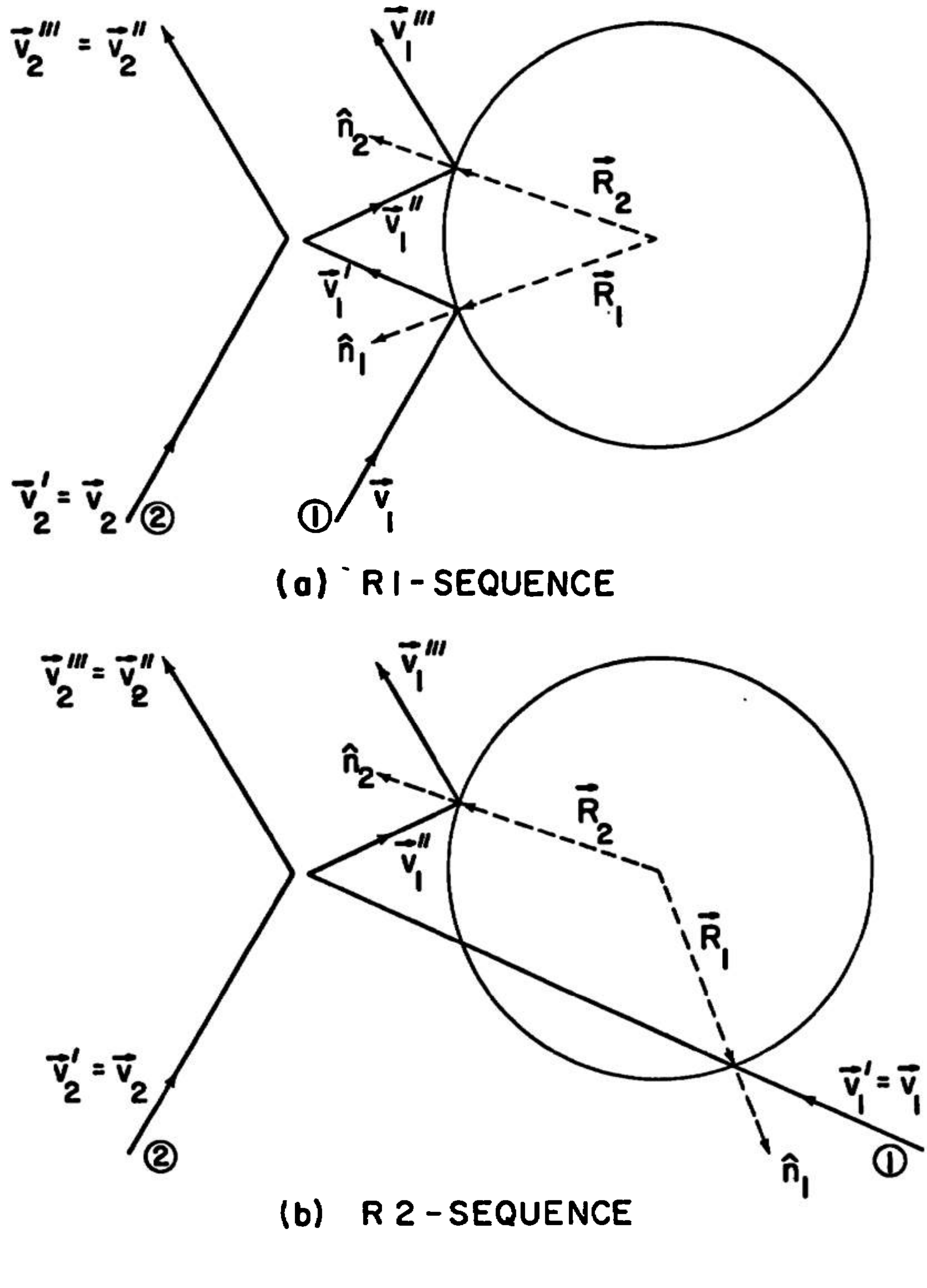


Figure 4. Recollisions

$$\vec{E}_{R1} = -mn^2\sigma^2\int d\vec{r}_1 d\vec{r}_2 d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{\vec{v}_1\cdot\hat{n}<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}_1| \; \delta^3(\vec{r}-\vec{R}_1)$$

$$\cdot \int d\vec{v}_1' \; \eta(\vec{v}_1'|\vec{v}_1) \int_{\vec{v}'_{12}\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \; |\vec{v}'_{12}\cdot\hat{\sigma}_{12}| \; \delta^3(\vec{r}_{12}+\vec{v}'_{12}\tau_1-\vec{\sigma}_{12})$$

$$\cdot \int_{\vec{v}_1''\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_1''\cdot\hat{n}_1| \; \delta^3(\vec{r}_1+\vec{v}_1'\tau_1+\vec{v}_1''\tau_2-\vec{R}_2) \int d\vec{v}_1''' \eta(\vec{v}_1'''|\vec{v}_1'')(\vec{v}_1'''-\vec{v}_1''), \quad (2\text{-}64)$$

where $d\vec{A}_1$ and $d\vec{A}_2$ are the surface elements at the positions $\vec{R}_1$ and $\vec{R}_2$, respectively. The time τ_1 is to be identified as the time between the first and the second collision and the time τ_2 as the time between the second and the third collision. The first two delta functions in (2-64) can be readily integrated and we obtain [19]

$$\vec{E}_{R1} = -mn^2\sigma^2\int d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2$$

$$\cdot \int_{\vec{v}_1\cdot\hat{n}_1<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}_1| \int d\vec{v}_1' \; \eta(\vec{v}_1'|\vec{v}_1) \int_{\vec{v}'_{12}\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} |\vec{v}'_{12}\cdot\hat{\sigma}_{12}| \int_{\vec{v}_1''\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_1''\cdot\hat{n}_2|$$

$$\cdot \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1'\tau_1+\vec{v}_1''\tau_2) \int d\vec{v}_1''' \eta(\vec{v}_1'''|\vec{v}_1'')(\vec{v}_1'''-\vec{v}_1''). \quad (2\text{-}65a)$$

Taking cognizance of the difference between $T^i(1X)$ and $T^n(1X)$, we obtain for $\vec{E}_{R2}$

$$\vec{E}_{R2} = +mn^2\sigma^2\int d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2$$

$$\cdot \int_{\vec{v}_1\cdot\hat{n}_1<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}_1| \int_{\vec{v}_{12}\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \; |\vec{v}_{12}\cdot\hat{\sigma}_{12}| \int_{\vec{v}_1''\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_1''\cdot\hat{n}_2| \; \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1\tau_1+\vec{v}_1''\tau_2)$$

$$\cdot \int d\vec{v}_1''' \eta(\vec{v}_1'''|\vec{v}_1'')(\vec{v}_1'''-\vec{v}_1''). \quad (2\text{-}65b)$$

Following the same procedure we obtain for the contributions $\vec{E}_{C1}$ and $\vec{E}_{C2}$ from cyclic collisions

$$\vec{E}_{C1} = -mn^2\sigma^2 \int d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2$$

$$\cdot \int_{\vec{v}_1\cdot\hat{n}_1<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}_1| \int d\vec{v}_1' \eta(\vec{v}_1'|\vec{v}_1) \int_{\vec{v}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \; |\vec{v}_{12}'\cdot\hat{\sigma}_{12}| \int_{\vec{v}_2''\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_2''\cdot\hat{n}_2|$$

$$\cdot \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1'\tau_1+\vec{v}_2''\tau_2) \int d\vec{v}_2''' \eta(\vec{v}_2'''|\vec{v}_2'')(\vec{v}_2'''-\vec{v}_2'') \;, \qquad \text{(2-66a)}$$

$$\vec{E}_{C2} = +mn^2\sigma^2 \int d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2$$

$$\cdot \int_{\vec{v}_1\cdot\hat{n}_1<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}_1| \int_{\vec{v}_{12}\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \; |\vec{v}_{12}\cdot\sigma_{12}| \int_{\vec{v}_2''\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_2''\cdot\hat{n}_2| \; \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1\tau_1+\vec{v}_2'\tau_2)$$

$$\cdot \int d\vec{v}_2''' \eta(\vec{v}_2'''|\vec{v}_2'')(\vec{v}_2'''-\vec{v}_2'') \;, \qquad \text{(2-66b)}$$

where the various velocities are now indicated in Fig. 5 for which

$$\vec{v}_2' = \vec{v}_2, \; \vec{v}_1''' = \vec{v}_1'' \qquad \text{(C1-sequence)}, \qquad \text{(2-67a)}$$

and

$$\vec{v}_1' = \vec{v}_1, \; \vec{v}_2' = \vec{v}_2, \; \vec{v}_1''' = \vec{v}_1'' \qquad \text{(C2-sequence)} \;. \qquad \text{(2-67b)}$$

The contributions $\vec{E}_{H1}$, and $\vec{E}_{H2}$ from hypothetical collisions are

$$\vec{E}_{H1} = +mn^2\sigma^2 \int d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2$$

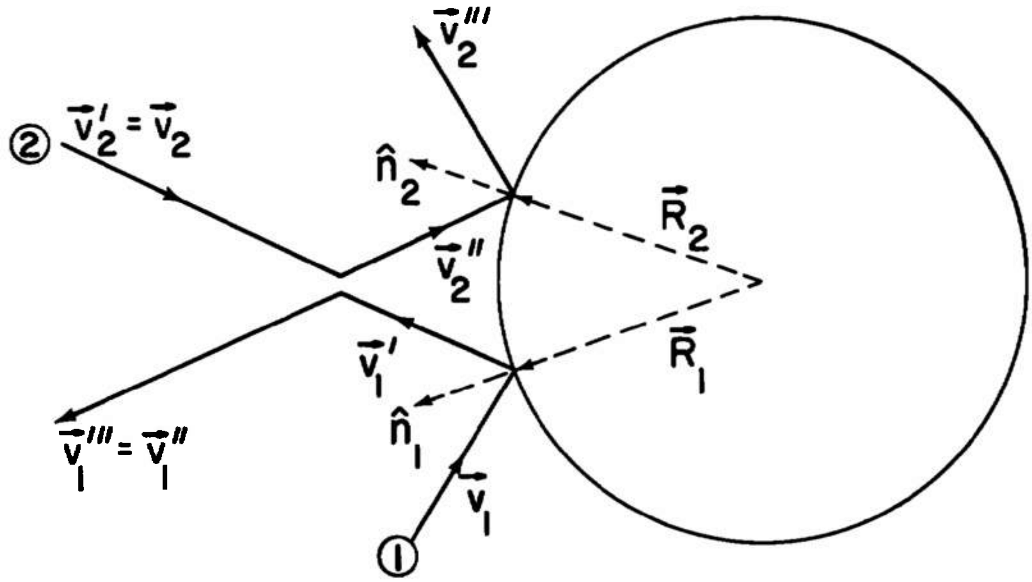
$$\cdot \int_{\vec{v}_1\cdot\hat{n}_1<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}_1| \int d\vec{v}_1' \eta(\vec{v}_1'|\vec{v}_1) \int_{\vec{v}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} |\vec{v}_{12}'\cdot\hat{\sigma}_{12}| \int_{\vec{v}_2\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_2\cdot\hat{n}_2|$$

$$\cdot \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1'\tau_1+\vec{v}_2\tau_2) \int d\vec{v}_2''' \eta(\vec{v}_2'''|\vec{v}_2'')(\vec{v}_2'''-\vec{v}_2'') \qquad \text{(2-68a)}$$

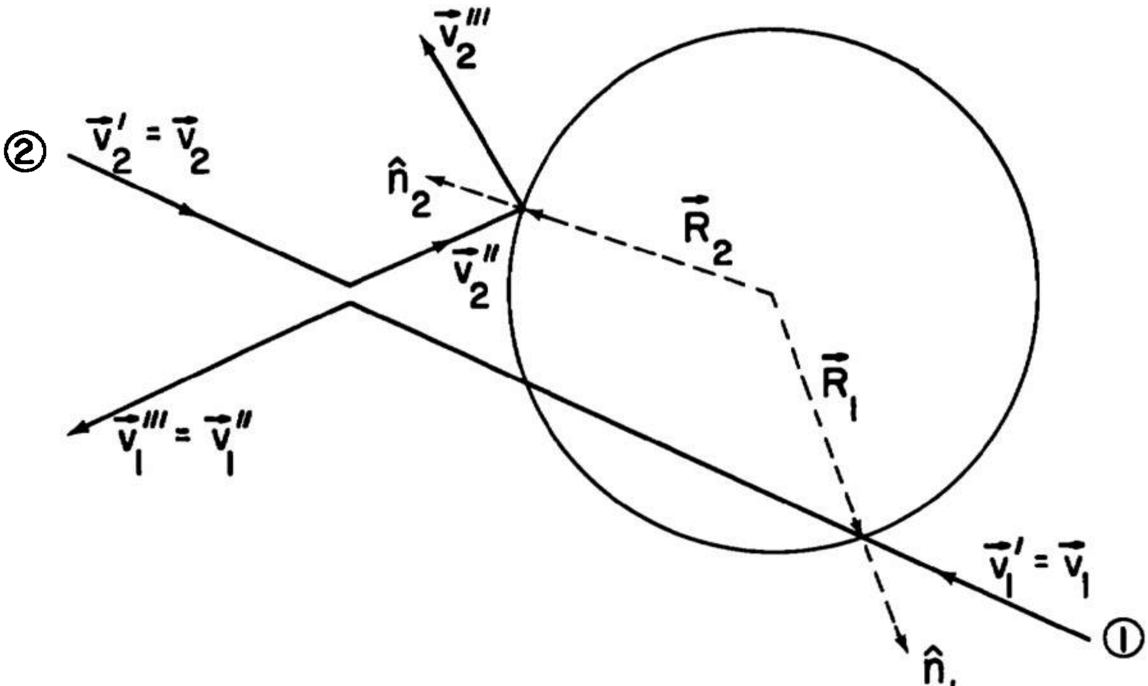
$$\vec{E}_{H2} = -mn^2\sigma^2 \int d\vec{v}_1 d\vec{v}_2 \; f(\vec{v}_1;\vec{V}) f(\vec{v}_2;\vec{V}) \int_0^\infty d\tau_1 \int_0^\infty d\tau_2$$

$$\cdot \int_{\vec{v}_1\cdot\hat{n}_1<0} d\vec{A}_1 \; |\vec{v}_1\cdot\hat{n}| \int_{\vec{v}_{12}\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \; |\vec{v}_{12}\cdot\hat{\sigma}_{12}| \int_{\vec{v}_2\cdot\hat{n}_2<0} d\vec{A}_2 \; |\vec{v}_2\cdot\hat{n}_2| \; \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1\tau_1+\vec{v}_2\tau_2)$$

$$\cdot \int d\vec{v}_2''' \eta(\vec{v}_2'''|\vec{v}_2'')(\vec{v}_2'''-\vec{v}_2'') \qquad \text{(2-68b)}$$

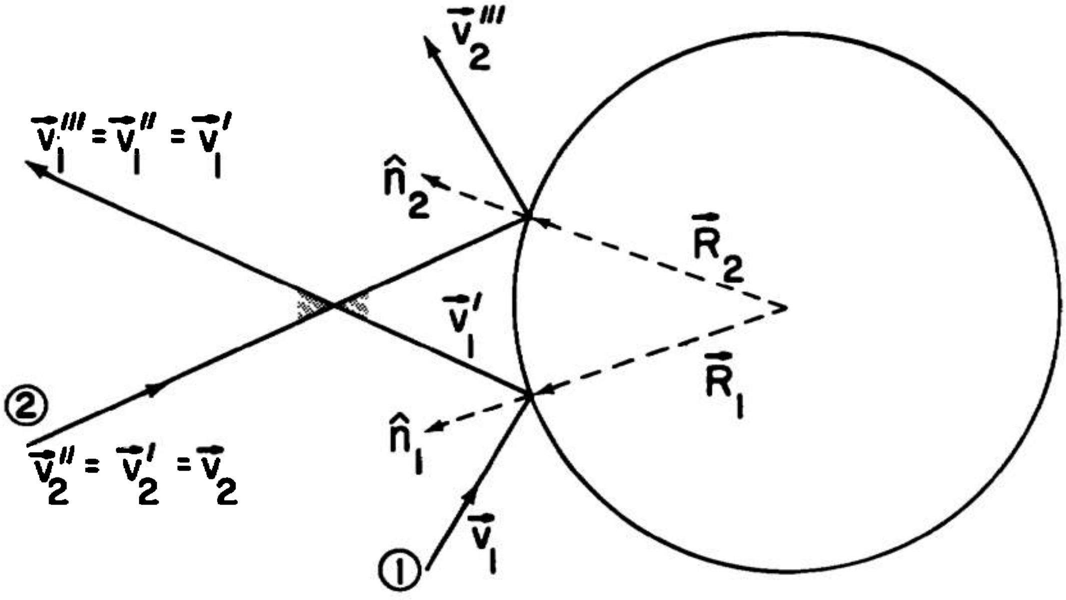


(a) C1 - SEQUENCE

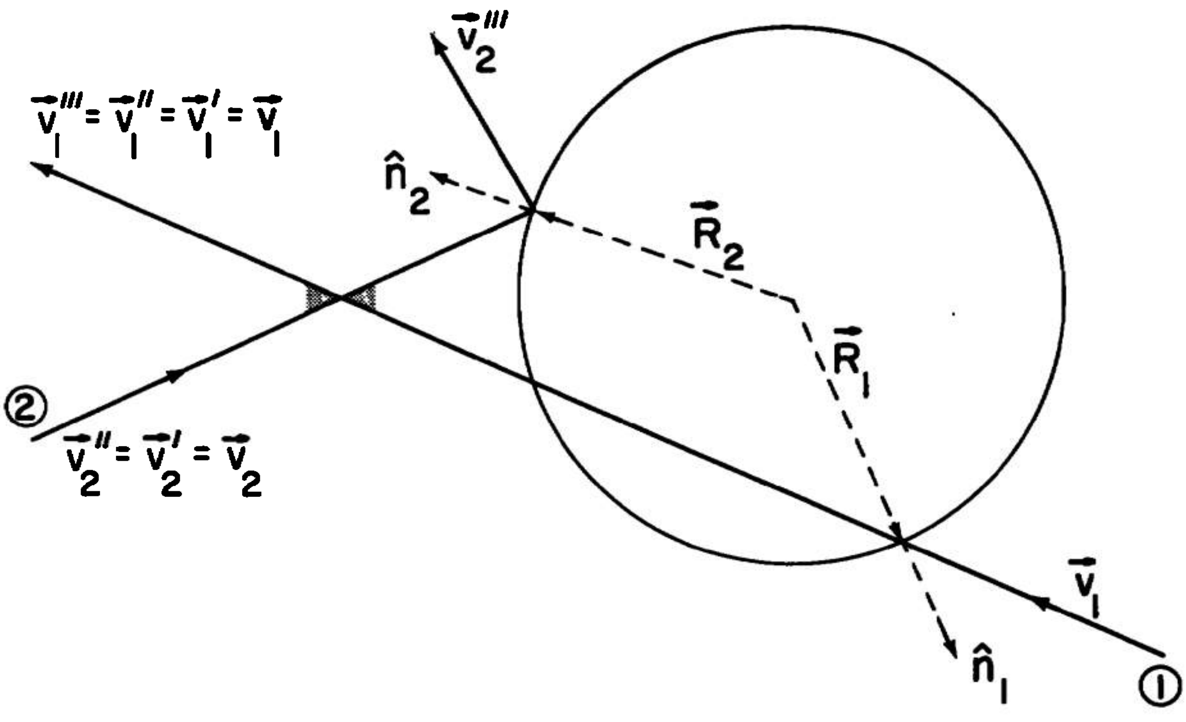


(b) C2 - SEQUENCE

Figure 5. Cyclic collisions.



(a) H1-SEQUENCE



(b) H2-SEQUENCE

Figure 6. Hypothetical collisions.

where the various velocities are now indicated in Fig. 6, for which

$$\vec{v}_2''=\vec{v}_2'=\vec{v}_2,\ \vec{v}_1'''=\vec{v}_1''=\vec{v}_1' \qquad \text{(H1-sequence)}, \qquad (2\text{-}69a)$$

and

$$\vec{v}_2''=\vec{v}_2'=\vec{v}_2,\ \vec{v}_1'''=\vec{v}_1''=\vec{v}_1'=\vec{v}_1 \qquad \text{(H2-sequence)}. \qquad (2\text{-}69b)$$

If the molecules of the gas are approximated by hard spheres a major simplification occurs, since we can prove that the recollisions and the cyclic collisions yield identical contributions

$$\vec{E}_{C1}=\vec{E}_{R1}\ ,\ \vec{E}_{C2}=\vec{E}_{R2} \qquad (2\text{-}70)$$

This theorem is proved in the Appendix.

In conclusion, we may approximate the first inverse Knudsen number correction to the drag force by

$$\vec{E}_1=\vec{E}_{H1}+\vec{E}_{H2}+2(\vec{E}_{R1}+\vec{E}_{R2})\ , \qquad (2\text{-}71)$$

with $\vec{E}_{R1}$ and $\vec{E}_{R2}$ given by (2-65) and $\vec{E}_{H1}$ and $\vec{E}_{H2}$ given by (2-68). The term $\vec{E}_{H1}$ represents a "loss" term which accounts for the fact that molecules reflected from the object prevent some molecules of the incident beam from striking the object. The term $2\vec{E}_{R1}$ represents a "gain" term which accounts for the fact that molecules reflected from the object cause some additional molecules to collide with the object. The terms $\vec{E}_{H2}$ and $2\vec{E}_{R2}$ account for a perturbation of the velocity distribution as a result of the fact that the presence of the object leads to a region that is inaccessible to some gas molecules. The terms $\vec{E}_{H2}$ and $2\vec{E}_{R2}$ vanish in the infinite Mach number limit [19], but need to be included in determining the drag force at small speed ratios.

CHAPTER III

DRAG COEFFICIENT OF A DISC IN THE NEARLY FREE MOLECULAR FLOW REGIME

3.1 Introduction

As a first application of our method we consider the drag force exerted on a disc. A study of the drag of a disc became very appealing, since, just prior to this research effort, Willis had obtained an analytic solution for this case in the infinite Mach number limit [20,21]†

The situation is illustrated in Fig. 7. A disc with radius R is placed perpendicular to a gas stream with flow velocity $\vec{V}$. The disc is located in the XY-plane and the Z-axis is taken in the direction of $-\vec{V}$.

The distribution of the velocities of the molecules in the gas stream can be represented by a displaced Maxwell distribution (2-57)

$$f(\vec{v}_i;\vec{V}) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left\{-\frac{m(\vec{v}_i-\vec{V})^2}{2kT}\right\} . \tag{3-1}$$

The molecules are assumed to be reflected diffusively at the surface so that the transition probability $\eta(\vec{v}'|\vec{v})$ in (2-15) is given by

$$\eta(\vec{v}'|\vec{v}) \equiv \eta(\vec{v}') = \frac{1}{2\pi}\left(\frac{m}{kT}\right)^2 (\vec{v}'\cdot\hat{n})\, e^{-\frac{mv'^2}{2kT}} \, \Theta(\vec{v}'\cdot\hat{n}) \, , \tag{3-2}$$

† We are indebted to Dr. A. G. Keel of the Naval Ordnance Laboratory for informing us about this work prior to publication.

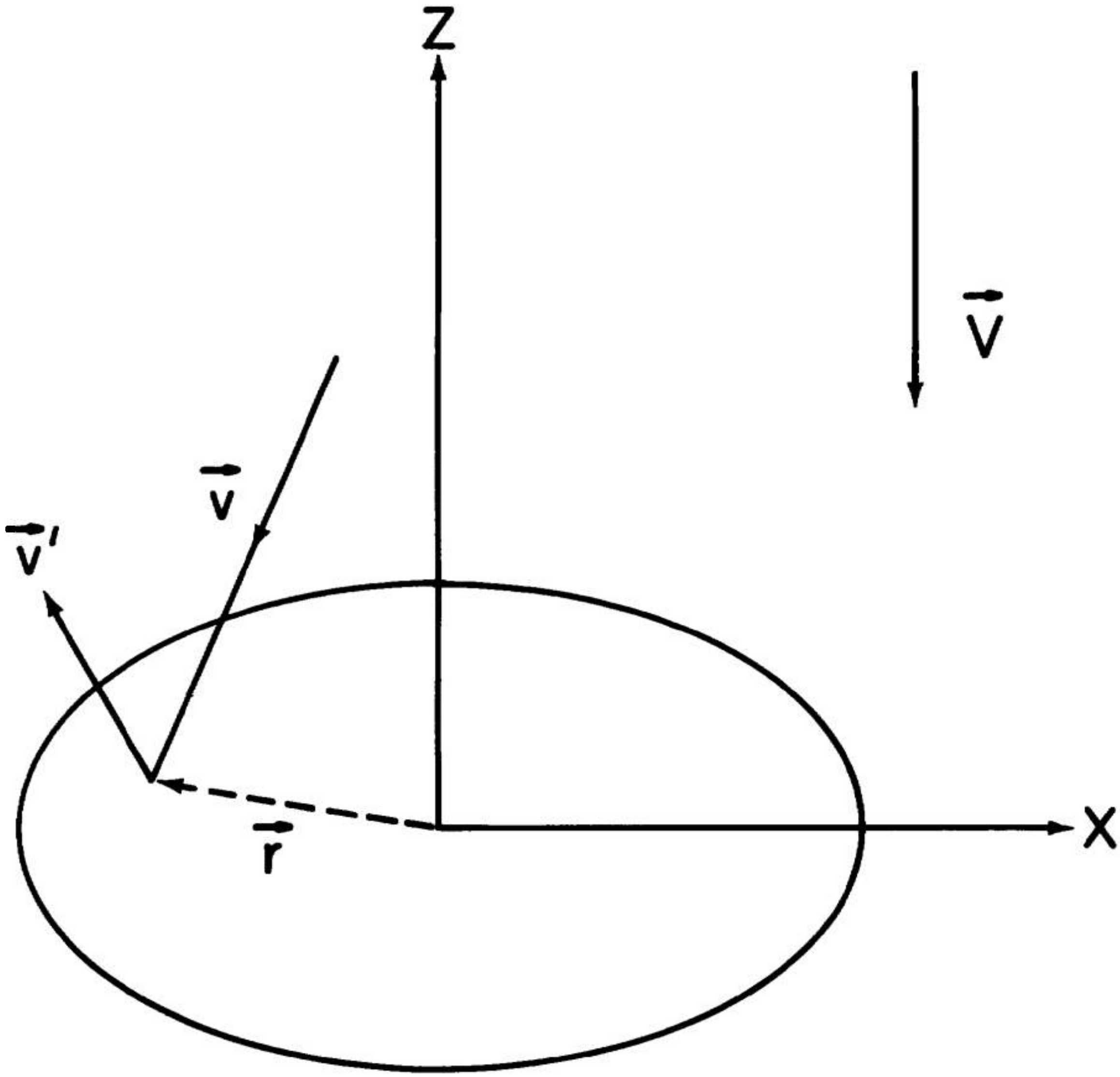


Figure 7. A disc placed in a gas stream with flow velocity $\vec{V}$.

where Θ is a Heaviside function defined as

$$\Theta(x) = 1 \qquad \text{for } x > o \ , \tag{3-3}$$

$$\Theta(x) = 0 \qquad \text{for } x < o \ .$$

The velocities $\vec{v}$ and $\vec{v}'$ are the velocities of a molecule before and after striking the object. The vector $\hat{n}$ is the normal vector of the object surface. At the upper side of the disc surface $\hat{n}$ is the unit vector in the +Z-direction; at the lower side of the disc surface $\hat{n}$ is the unit vector in the -Z-direction. The temperature T in (3-2) is the temperature of the object which is taken to be equal to the temperature T of the gas in (3-1).

In order to exhibit the dependence of the drag force on the inverse Knudsen number explicitly it is convenient to introduce dimensionless quantities. For this purpose we measure all distances in terms of the radius R of the disc and all velocities in terms of the thermal velocity $(2kT/m)^{1/2}$ of the molecules. We thus define

$$\vec{w}_i = \vec{v}_i \left(\frac{m}{2kT}\right)^{1/2}, \quad \vec{S} = \vec{V}\left(\frac{m}{2kT}\right)^{1/2}, \quad \tau^* = \frac{\tau}{R}\left(\frac{2kT}{m}\right)^{1/2} . \tag{3-4}$$

The magnitude of the dimensionless flow velocity $\vec{S}$ is the speed ratio earlier introduced in (1-5). The surface element $d\vec{A}$ of the disc surface can be written as

$$d\vec{A} = R^2 d\vec{r} \ , \tag{3-5}$$

where $\vec{r}$ is a two dimensional vector in the plane of the disc such that $o<r<1$. In addition we introduce a dimensionless distribution function $f^*(\vec{w}_i;\vec{S})$ and a dimensionless transition probability $\eta^*(\vec{w}_i')$ defined as

$$f^*(\vec{w}_i;\vec{S}) = \frac{1}{\pi^{3/2}} e^{-(\vec{w}_i - \vec{S})^2} \ , \tag{3-6}$$

$$\eta^*(\vec{w}_i') = \frac{2}{\pi} (\vec{w}_i' \cdot \hat{n}) e^{-w_i'^2} \Theta(\vec{w}_i' \cdot \hat{n}) \quad , \tag{3-7}$$

normalized such that $\int d\vec{w}_i \; f^*(\vec{w}_i; \vec{S}) = 1$ and $\int d\vec{w}_i' \; \eta^*(\vec{w}_i') = 1$.

The kinetic energy of the gas stream is $U_K = \frac{1}{2} nmV^2 \pi R^2$, so that, in accordance with (1-1), the drag coefficient of a disc with radius R is defined as

$$C_D = \frac{2E}{nmV^2 \pi R^2} \quad . \tag{3-8}$$

As a result of symmetry the drag force has only a non-vanishing component in the direction of the flow velocity $\vec{V}$, so that

$$E = \vec{E} \cdot \hat{V} \quad . \tag{3-9}$$

In this report we define the Knudsen number K as the ratio of the mean free path λ over the radius R of the disc. Since for a gas of hard spheres $\lambda^{-1} = \sqrt{2} \pi n \sigma^2$ [3], we thus define

$$\boxed{K^{-1} = \sqrt{2} \; \pi n \sigma^2 R.} \tag{3-10}$$

The drag coefficient C_D in the nearly free molecular flow regime can be written as

$$C_D = C_o + C_1 K^{-1} + \ldots \quad . \tag{3-11}$$

The drag coefficient C_o in the free molecular flow limit $K^{-1} \to o$ follows from (2-56)

$$C_o = -\frac{2}{\pi S^2} \int d\vec{w} \; f^*(\vec{w}; \vec{S}) \int_{\vec{w} \cdot \hat{n} < o} d\hat{r} \; |\vec{w} \cdot \hat{n}| \int d\vec{w}' \eta^*(\vec{w}') (\vec{w}' - \vec{W}) \cdot \hat{S}. \tag{3-12}$$

The coefficient C_1 of the first inverse Knudsen number correction can be decomposed in analogy to (2-71)

$$C_1 = C_{H1} + C_{H2} + 2(C_{R1} + C_{R2}) \quad . \tag{3-13}$$

Table I.

Collision integrals for the first inverse Knudsen number correction to the drag coefficient of a disc.

$$C_{H1} = + \frac{\sqrt{2}}{\pi^2 S^2} \int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_0^\infty d\tau_2^* \int_{\vec{w}_1\cdot\hat{n}<o} d\vec{r}_1 |\vec{w}_1\cdot\hat{n}_1| \int d\vec{w}_1' \, \eta^*(\vec{w}_1')$$

$$\cdot \int_{\vec{w}_{12}'\cdot\hat{\sigma}_{12}<o} d\hat{\sigma}_{12} |\vec{w}_{12}'\cdot\hat{\sigma}_{12}| \int_{\vec{w}_2\cdot\hat{n}_2<o} d\vec{r}_2 |\vec{w}_2\cdot\hat{n}_2| \delta^3(\vec{r}_1-\vec{r}_2+\vec{w}_1'\tau_1^*+\vec{w}_2\tau_2^*) \int d\vec{w}_2''' \eta^*(\vec{w}_2''')(\vec{w}_2'''-\vec{w}_2).\hat{S}, \quad \text{(I-1)}$$

$$C_{H2} = - \frac{\sqrt{2}}{\pi^2 S^2} \int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_0^\infty d\tau_2^* \int_{\vec{w}_1\cdot\hat{n}_1<o} d\vec{r}_1 |\vec{w}_1\cdot\hat{n}_1|$$

$$\cdot \int_{\vec{w}_{12}\cdot\hat{\sigma}_{12}<o} d\hat{\sigma}_{12} |\vec{w}_{12}\cdot\hat{\sigma}_{12}| \int_{\vec{w}_2\cdot\hat{n}_2<o} d\vec{r}_2 |\vec{w}_2\cdot\hat{n}_2| \delta^3(\vec{r}_1-\vec{r}_2+\vec{w}_1\tau_1^*+\vec{w}_2\tau_2^*) \int d\vec{w}_2''' \eta^*(\vec{w}_2''')(\vec{w}_2'''-\vec{w}_2).\hat{S}. \quad \text{(I-2)}$$

$$C_{R1} = - \frac{\sqrt{2}}{\pi^2 S^2} \int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_0^\infty d\tau_2^* \int_{\vec{w}_1\cdot\hat{n}_1<o} d\vec{r}_1 |\vec{w}_1\cdot\hat{n}_1| \int d\vec{w}_1' \, \eta^*(\vec{w}_1')$$

$$\cdot \int_{\vec{w}_{12}'\cdot\hat{\sigma}_{12}<o} d\hat{\sigma}_{12} |\vec{w}_{12}'\cdot\hat{\sigma}_{12}| \int_{\vec{w}_1''\cdot\hat{n}_2<o} d\vec{r}_2 |\vec{w}_1''\cdot\hat{n}_2| \delta^3(\vec{r}_1-\vec{r}_2+\vec{w}_1'\tau_1^*+ \vec{w}_1''\tau_2^*) \int d\vec{w}_1''' \eta^*(\vec{w}_1''')(\vec{w}_1'''-\vec{w}_1'').\hat{S}, \quad \text{(I-3)}$$

$$C_{R2} = + \frac{\sqrt{2}}{\pi^2 S^2} \int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_0^\infty d\tau_2^* \int_{\vec{w}_1\cdot\hat{n}_1<o} d\vec{r}_1 |\vec{w}_1\cdot\hat{n}_1|$$

$$\cdot \int_{\vec{w}_{12}\cdot\hat{\sigma}_{12}<o} d\hat{\sigma}_{12} |\vec{w}_{12}\cdot\hat{\sigma}_{12}| \int_{\vec{w}_1''\cdot\hat{n}_2<o} d\vec{r}_2 |\vec{w}_1''\cdot\hat{n}_2| \delta^3(\vec{r}_1-\vec{r}_2+\vec{w}_1\tau_1^*+\vec{w}_1''\tau_2^*) \int d\vec{w}_1''' \eta^*(\vec{w}_1''')(\vec{w}_1'''\cdot\vec{w}_1'').\hat{S}. \quad \text{(I-4)}$$

Note: $f^*(\vec{w}_i\cdot\vec{S}) = \pi^{-3/2} e^{-(\vec{w}_i-\vec{S})^2}$,

$\eta^*(\vec{w}_i') = \frac{2}{\pi}(\vec{w}_i'\cdot\hat{n}) e^{-w_i'^2} \Theta(\vec{w}_i'\cdot\hat{n})$ (diffusive reflection).

For a sphere: $\vec{r}_1 = \hat{n}_1 = \hat{R}_1$, $\vec{r}_2 = \hat{n}_2 = \hat{R}_2$.

The explicit expressions for the collision integrals C_{H1}, C_{H2}, C_{R1}, and C_{R2} follow from (2-68) and (2-65) and are listed in Table I. The velocities in the integrands refer again to the hypothetical collisions and recollisions which, for a disc, are shown schematically in Figs. 8 and 9. We note that all collision integrals contain a factor of the type $\int d\vec{w}'\eta^*(\vec{w}')(\vec{w}'-\vec{w})$ which for diffusive reflection reduces to

$$\int d\vec{w}'\eta^*(\vec{w}')(\vec{w}'-\vec{w}) = \frac{2}{\pi}\int_{\vec{w}'\cdot\hat{n}>0} d\vec{w}' e^{-w'^2}(\vec{w}'\cdot\hat{n})(\vec{w}'-\vec{w}) = \frac{\sqrt{\pi}}{2}\hat{n} - \vec{w} \,. \tag{3-14}$$

The drag coefficient C_D is a function of the speed ratio S. For a disc it may be decomposed into a contribution C_D^+ accounting for the force exerted at the upper side of the disc surface and a contribution C_D^- accounting for the force exerted at the lower side of the disc surface

$$C_D = C_D^+ + C_D^- \,. \tag{3-15}$$

The two contributions are interrelated by

$$C_D^-(+S) = -C_D^+(-S) \quad . \tag{3-16}$$

3.2 Drag coefficient C_o of a Disc in the Free Molecular Flow Regime.

The drag coefficient C_o of the disc in the free molecular flow regime is given by (3-12) which can be integrated over the area of the disc. The drag coefficient C_o of a disc is thus the same as the coefficient C_o of a flat plate of any shape. We decompose C_o in analogy to (3-15) and (3-16)

$$C_o(S) = C_o^+(S) + C_o^-(S) = C_o^+(S) - C_o^+(-S) \;, \tag{3-17}$$

and note that

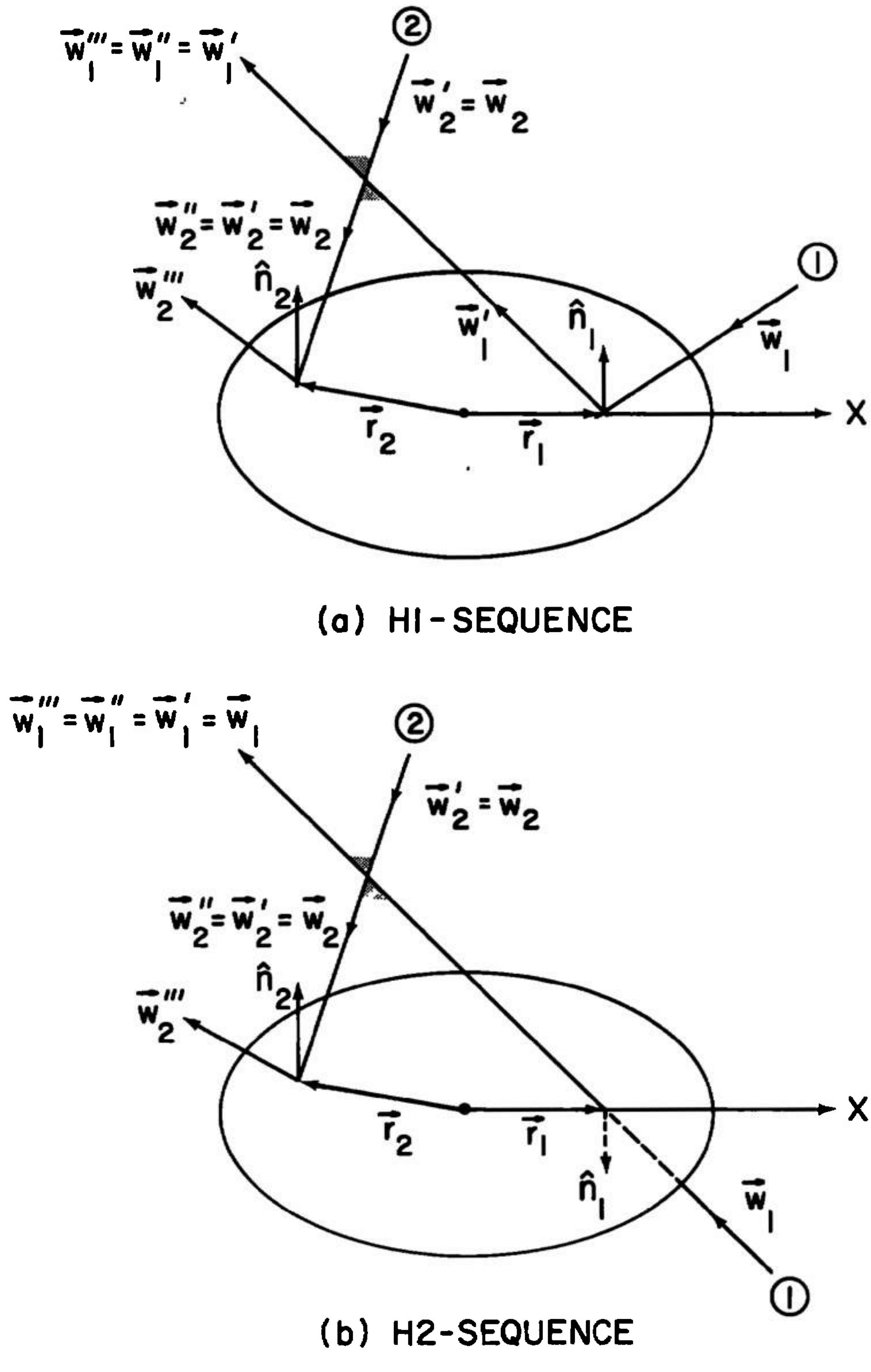


Figure 8. Hypothetical collisions for a disc.

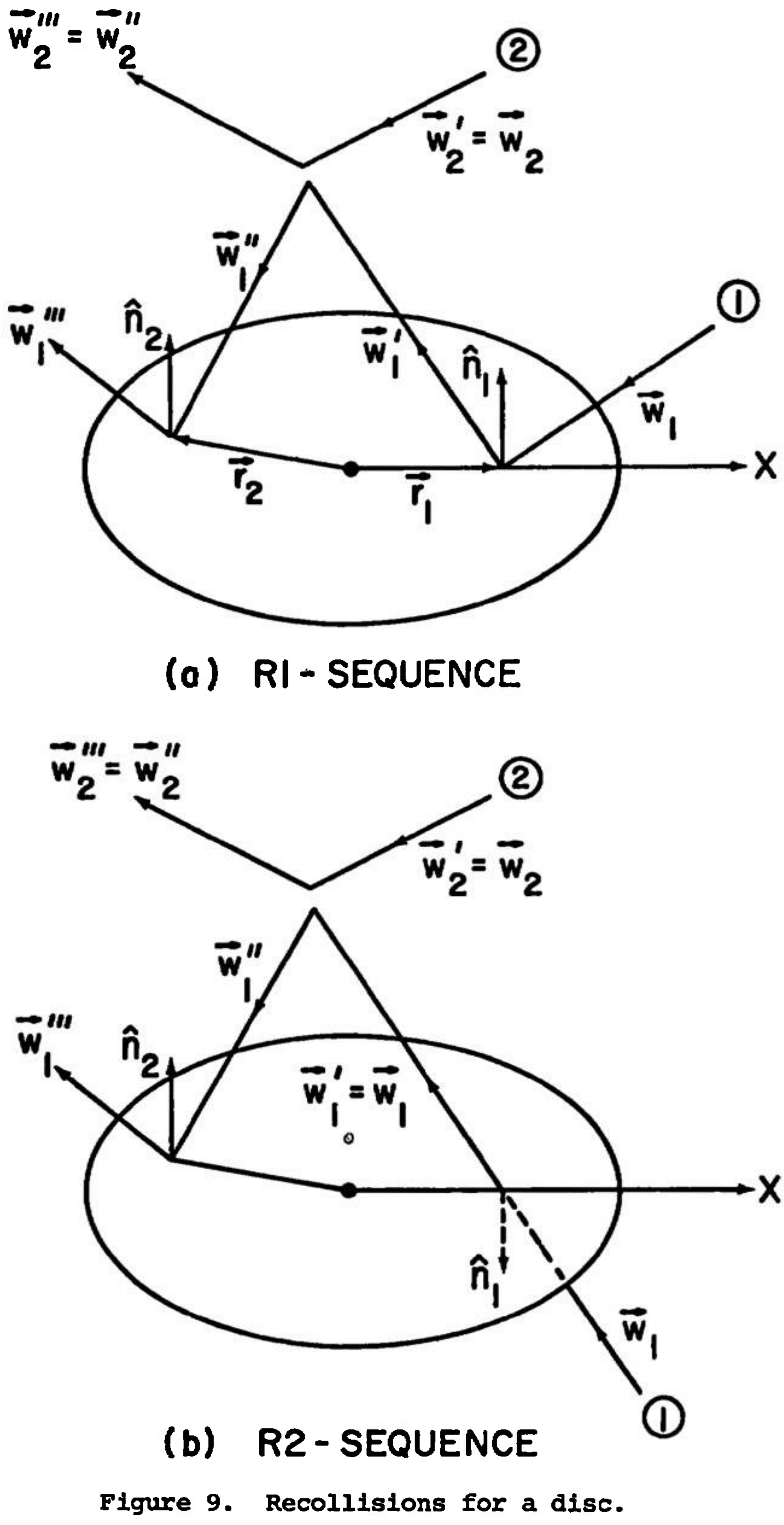


Figure 9. Recollisions for a disc.

$$C_o^+(S) = \frac{2}{\pi^{3/2} S^2} \int_{w_z<o} d\vec{w}\, e^{-(\vec{w}-\vec{S})^2} \left(w_z^2 - \frac{\sqrt{\pi}}{2} w_z\right) = \tag{3-18}$$

$$= \frac{1}{\sqrt{\pi} S^2} \left[S e^{-S^2} + \frac{\sqrt{\pi}}{2}(1+2S^2)(1+\mathrm{erf}\, S) + \frac{\sqrt{\pi}}{2} \{ e^{-S^2} + S\sqrt{\pi}(1+\mathrm{erf}\, S) \} \right] ,$$

where erf S is the error function defined as

$$\mathrm{erf}\, S \equiv \frac{2}{\sqrt{\pi}} \int_o^S e^{-t^2} dt \quad . \tag{3-19}$$

We thus obtain for the total free molecular drag coefficient C_o

$$\boxed{C_o(S) = \frac{2}{S\sqrt{\pi}} \left[e^{-S^2} + \sqrt{\pi}\left(\frac{1}{2S} + S\right) \mathrm{erf}\, S + \frac{\pi}{2} \right]} \quad . \tag{3-20}$$

This expression is in agreement with the result obtained by previous investigators [32], but it has not been reproduced correctly in some more popular reviews [1,2,10].

In the low and high speed limits the drag coefficient $C_o(S)$ reduces to

$$\lim_{S\to o} C_o(S) = \frac{(4+\pi)}{\sqrt{\pi}} \cdot \frac{1}{S} = \frac{4.029}{S} \quad , \tag{3-21}$$

and

$$\lim_{S\to\infty} C_o(S) = 2 \quad . \tag{3-22}$$

Thus at low speed ratios the drag coefficient $C_o(S)$ becomes inversely proportional to the speed ratio S, while in the high speed limit the drag coefficient $C_o(S)$ becomes independent of the speed ratio S. On comparing with (3-8) we note that this result means that at low velocities the force is proportional to the stream velocity, as is to be expected [16], while at high velocities the force increases with the square of the stream velocity.

3.3 Reduction of Collision Integrals for C_1

The coefficient C_1 of the first inverse Knudsen number correction to the drag coefficient is determined by the collision integrals in Table I. Since

$$C_1(S) = C_1^+(S) - C_1^+(-S), \tag{3-23}$$

in accordance with (3-15) and (3-16), we only need to analyze the collision integrals for

$$C_1^+ = C_{H1}^+ + C_{H2}^+ + 2(C_{R1}^+ + C_{R2}^+). \tag{3-24}$$

These collision integrals correspond to the case that the normal vector $\hat{n}_2$ is in the + Z-direction, i.e. in the direction of $-\hat{S}$.

It is convenient to introduce a set of auxiliary vectors defined as

$$\vec{w}_{H1} \equiv \frac{\vec{w}_1'}{\vec{w}_1' \cdot \hat{n}_1} - \frac{\vec{w}_2}{\vec{w}_2 \cdot \hat{n}_2} = \frac{\vec{w}_1'}{w_{1z}'} - \frac{\vec{w}_2}{w_{2z}}, \tag{3-25a}$$

$$\vec{w}_{H2} \equiv \frac{\vec{w}_1}{|\vec{w}_1 \cdot \hat{n}_1|} - \frac{\vec{w}_2}{\vec{w}_2 \cdot \hat{n}_2} = \frac{\vec{w}_1}{w_{1z}} - \frac{\vec{w}_2}{w_{2z}}, \tag{3-25b}$$

$$\vec{w}_{R1} \equiv \frac{\vec{w}_1'}{\vec{w}_1' \cdot \hat{n}_1} - \frac{\vec{w}_1''}{\vec{w}_1'' \cdot \hat{n}_2} = \frac{\vec{w}_1'}{w_{1z}'} - \frac{\vec{w}_1''}{w_{1z}''}, \tag{3-25c}$$

$$\vec{w}_{R2} \equiv \frac{\vec{w}_1}{|\vec{w}_1 \cdot \hat{n}_1|} - \frac{\vec{w}_1''}{\vec{w}_1'' \cdot \hat{n}_2} = \frac{\vec{w}_1}{w_{1z}} - \frac{\vec{w}_1''}{w_{1z}''}. \tag{3-25d}$$

These auxiliary vectors are two-dimensional vectors located in the plane of the disc. The normal vector $\hat{n}_1$ is directed in the positive Z-direction for the integrals associated with the H1- and R1- sequences and in the negative Z-direction for the integrals associated with the H2- and R2- sequences. For future reference we also define

$$I(\vec{w}_{H1}) \equiv \frac{w_{H1}}{(\vec{w}_{H1} \times \hat{r}_1)^2} - \frac{|\vec{w}_{H1}\cdot\hat{r}_1|^3}{w_{H1}^2(\vec{w}_{H1} \times \hat{r}_1)^2} - \frac{\vec{w}_{H1}\cdot\hat{r}_1}{w_{H1}^2} , \tag{3-26a}$$

$$I(\vec{w}_{H2}) \equiv \frac{w_{H2}}{(\vec{w}_{H2} \times \hat{r}_1)^2} - \frac{|\vec{w}_{H2}\cdot\hat{r}_1|^3}{w_{H2}^2(\vec{w}_{H2} \times \hat{r}_1)^2} - \frac{\vec{w}_{H2}\cdot\hat{r}_1}{w_{H2}^2} , \tag{3-26b}$$

$$I(\vec{w}_{R1}) \equiv \frac{w_{R1}}{(\vec{w}_{R1} \times \hat{r}_1)^2} - \frac{|\vec{w}_{R1}\cdot\hat{r}_1|^3}{w_{R1}^2(\vec{w}_{R1} \times \hat{r}_1)^2} - \frac{\vec{w}_{R1}\cdot\hat{r}_1}{w_{R1}^2} , \tag{3-26c}$$

$$I(\vec{w}_{R2}) \equiv \frac{w_{R2}}{(\vec{w}_{R2} \times \hat{r}_1)^2} - \frac{|\vec{w}_{R2}\cdot\hat{r}_1|^3}{w_{R2}^2(\vec{w}_{R2} \times \hat{r}_1)^2} - \frac{\vec{w}_{R2}\cdot\hat{r}_1}{w_{R2}^2} . \tag{3-26d}$$

As an example we consider the collision integral C_{H1}^{+} associated with the H1-sequences. The δ-function in (I-1) may be written

$$|\vec{w}_2\cdot\hat{n}_2|\,\delta^3(\vec{r}_1-\vec{r}_2+\vec{w}_1'\tau_1^*+\vec{w}_2\tau_2^*) = \delta\left(\tau_2^* + \frac{\vec{w}_1'\cdot\hat{n}_1}{\vec{w}_2\cdot\hat{n}_2}\tau_1^*\right)\delta^2(\vec{r}_1-\vec{r}_2+(\vec{w}_1'\cdot\hat{n}_1)\tau_1^*\vec{w}_{H1}) . \tag{3-27}$$

Because of symmetry we can integrate over the azimuthal angle of the position vector $\vec{r}_1$ of the first collision and take $\vec{r}_1$ in a fixed direction, say in the positive X-direction as indicated in Figs. 8 and 9. Using (3-14) and integrating over τ_2^* and $\vec{r}_2$ we thus obtain

$$\begin{aligned} C_{H1}^{+}(S) = -\frac{2\sqrt{2}}{\pi S^2}\int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_0^1 dr_1 r_1\, |w_{1z}| \\ \cdot \int d\vec{w}_1'\, \eta^*(\vec{w}_1') \int_{\vec{w}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} |\vec{w}_{12}'\cdot\hat{\sigma}_{12}|\, \left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) \Theta(-w_{1z})\, \Theta(-w_{2z}) \quad , \end{aligned} \tag{3-28}$$

with the auxiliary condition

$$r_2 = |\vec{r}_1 + (\vec{w}_1'\cdot\hat{n}_1)\tau_1^*\vec{w}_{H1}| \le 1 \quad . \tag{3-29}$$

This auxiliary condition implies

$$0 \le \tau_1^* \le \tau_{H1}^* = \frac{-\vec{w}_{H1}\cdot\hat{r}_1 + \sqrt{w_{H1}^2 - (\vec{w}_{H1} \times \hat{r}_1)^2}}{w_{H1}^2\, \vec{w}_1'\cdot\hat{n}_1} . \tag{3-30}$$

The time τ^*_{H1} is to be interpreted as the maximum possible time interval between the first and the second collision that leads to an H1-sequence for given $\vec{r}_1$, $\vec{w}'_1$ and $\vec{w}_2$. It then follows that

$$\int_0^\infty d\tau_1^* \int_0^1 dr_1 r_1\ \Theta(\tau^*_{H1} - \tau_1^*) = \frac{1}{3\vec{w}'_1\cdot\hat{n}_1}\ I(\vec{w}_{H1}), \tag{3-31}$$

with $I(\vec{w}_{H1})$ defined by (3-26a). Integration over $\hat{\sigma}_{12}$ yields

$$\int d\hat{\sigma}_{12}\ |\vec{w}'_{12}\cdot\hat{\sigma}_{12}|\ \Theta(-\vec{w}'_{12}\cdot\hat{\sigma}_{12}) = \pi w'_{12}\ , \tag{3-32}$$

and the collision integral (3-28) reduces to

$$C^+_{H1}(S) = -\frac{4\sqrt{2}}{3\pi S^2}\int d\vec{w}_1 d\vec{w}_2\ f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int d\vec{w}'_1 e^{-w_1'^2} |w_{1z}| \tag{3-33a}$$

$$\cdot\ w'_{12}\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H1})\ \Theta(-w_{1z})\ \Theta(-w_{2z})\ \Theta(w'_{1z}).$$

The other collision integrals can be treated in the same manner with the result

$$C^+_{H2}(S) = +\frac{2\sqrt{2}}{3S^2}\int d\vec{w}_1 d\vec{w}_2\ f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) w_{12}\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H2})\ \Theta(w_{1z})\ \Theta(-w_{2z}), \tag{3-33b}$$

$$C^+_{R1}(S) = +\frac{4\sqrt{2}}{3\pi S^2}\int d\vec{w}_1 d\vec{w}_2\ f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int d\vec{w}'_1 e^{-w_1'^2}\ |w_{1z}|$$

$$\cdot\ \frac{1}{\pi}\int d\hat{\sigma}_{12}\ |\vec{w}'_{12}\cdot\hat{\sigma}_{12}|\ \left(\frac{\sqrt{\pi}}{2} - w''_{1z}\right) I(\vec{w}_{R1})\ \Theta(-w_{1z})\ \Theta(w'_{1z})\ \Theta(-w''_{1z})\ \Theta(-\vec{w}'_{12}\cdot\hat{\sigma}_{12}), \tag{3-33c}$$

$$C^+_{R2}(S) = -\frac{2\sqrt{2}}{3S^2}\int d\vec{w}_1 d\vec{w}_2\ f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \frac{1}{\pi}\int d\hat{\sigma}_{12}\ |\vec{w}_{12}\cdot\hat{\sigma}_{12}|$$

$$\cdot\ \left(\frac{\sqrt{\pi}}{2} - w''_{1z}\right) I(\vec{w}_{R2})\ \Theta(w_{1z})\ \Theta(-w''_{1z})\ \Theta(-\vec{w}_{12}\cdot\hat{\sigma}_{12}). \tag{3-33d}$$

The total contribution of the four collision sequences to the coefficient C_1 is then determined by

$$C_{H1}(S)= C_{H1}^{+}(S)- C_{H1}^{+}(-S) \ , \tag{3-34a}$$

$$C_{H2}(S)= C_{H2}^{+}(S)- C_{H2}^{+}(-S) \ , \tag{3-34b}$$

$$C_{R1}(S)= C_{R1}^{+}(S)- C_{R1}^{+}(-S) \ , \tag{3-34c}$$

$$C_{R2}(S)= C_{R2}^{+}(S)- C_{R2}^{+}(-S) \ . \tag{3-34d}$$

From a computational point of view the major difference between C_{H1} and C_{H2} on the one hand and C_{R1} and C_{R2} on the other hand is that in the recollision integrals the integration over $\hat{\sigma}_{12}$ is to be done numerically, since the integrand depends on $\hat{\sigma}_{12}$ via

$$\vec{w}_1''= \vec{w}_1'- (\vec{w}_{12}'\cdot\hat{\sigma}_{12})\hat{\sigma}_{12}, \qquad \text{(in } C_{R1}) \tag{3-35a}$$

$$\vec{w}_1''= \vec{w}_1- (\vec{w}_{12}\cdot\hat{\sigma}_{12})\hat{\sigma}_{12}. \qquad \text{(in } C_{R2}) \tag{3-35b}$$

3.4 Drag Coefficient C_1 of a Disc at High Speed Ratios.

The calculation of the collision integrals for the coefficient C_1 becomes particularly simple when the disc is placed in a beam of molecules that move with uniform velocity $\vec{V}$. This limit is approximated when the velocity $\vec{V}$ of the gas stream is large compared to the thermal velocity $(2kT/m)^{1/2}$, i.e. $S >> 1$. We may refer to this limit as the high speed limit or (cold wall) beam limit.

In the beam limit all incident molecules are moving parallel, so that two molecules will never collide with each other unless at least one of them first interacts with the object. The contributions $\lim_{S\to\infty} C_{H2}(S)$ and $\lim_{S\to\infty} C_{R2}(S)$, therefore, vanish in the beam limit. Moreover, in the beam limit the drag force completely originates from collisions at the upper surface of the disc, so that

$$\lim_{S\to\infty} C_1(S)= \lim_{S\to\infty} C_{H1}^{+}(S)+ 2 \lim_{S\to\infty} C_{R1}^{+}(S). \tag{3-36}$$

The collision integrals for the drag coefficient in the beam limit are obtained when the distribution function $f^*(\vec{w}_i;\vec{S})$ is approximated by a δ-function

$$f^*(\vec{w}_i;\vec{S}) = \delta^3(\vec{w}_i - \vec{S}) . \tag{3-37}$$

Noting that $\vec{w}_1 = \vec{S}$, $\vec{w}_2 = \vec{S}$, $\vec{w}'_{12} \simeq -\vec{S}$, $\vec{w}''_1 = (\vec{S}\cdot\hat{\sigma}_{12})\hat{\sigma}_{12}$ we thus obtain from (3-33a) and (3-33c)

$$\lim_{S\to\infty} C_{H1}(S) = -\frac{4\sqrt{2}}{3\pi} S \int d\vec{w}'_1 e^{-w_1'^2} I(\vec{w}_{H1})\, \Theta(\vec{w}'_{1z}) , \tag{3-38a}$$

with

$$\vec{w}_{H1} = \frac{\vec{w}'_1}{w'_{1z}} + \hat{S} , \tag{3-38b}$$

and

$$\lim_{S\to\infty} C_{R1}(S) = +\frac{4\sqrt{2}}{3\pi^2} S \int d\vec{w}'_1 e^{-w_1'^2} \int d\hat{\sigma}_{12} (\hat{S}\cdot\hat{\sigma}_{12})^3 I(\vec{w}_{R1})\, \Theta(w_{1z})\, \Theta(\hat{S}\cdot\hat{\sigma}_{12}) , \tag{3-39a}$$

with

$$\vec{w}_{R1} = \frac{\vec{w}'_1}{w'_{1z}} - \frac{\hat{\sigma}_{12}}{\sigma_{12z}} . \tag{3-39b}$$

These integrals can be readily evaluated analytically. For this purpose we introduce the coordinate transformation

$$d\vec{w}'_1 = w'^2_{1z}\, dw'_{1z}\, w_\nu dw_\nu d\phi ,$$

where $w_\nu = w_{H1}$ in (3-38) and $w_\nu = w_{R1}$ in (3-39) and where ϕ is the azimuthal angle of the two dimensional vector $\vec{w}_\nu$ with the X-axis as initial axis. In terms of these variables the quantities $I(\vec{w}_{H1})$ and $I(\vec{w}_{R1})$, defined in (3-26), reduce to

$$I(\vec{w}_\nu) = \frac{1}{w_\nu} \left[\frac{1-|\cos^3\phi|}{\sin^2\phi} - \cos\phi \right] . \tag{3-40}$$

The perihelion vector $\hat{\sigma}_{12}$ may be expressed in terms of its polar angle θ_σ with the Z-axis and its azimuthal angle ϕ_σ with the XZ-plane as initial plane. The collision integrals then become

$$\lim_{S\to\infty} C_{H1}(S) = -\frac{4\sqrt{2}}{3\pi} S \int_0^\infty dw'_{1z}\, w'^2_{1z} \int_0^\infty dw_{H1} e^{-w'^2_{1z}(w^2_{H1}+1)}$$

$$\cdot \int_0^{2\pi} d\phi \left[\frac{1-|\cos^3\phi|}{\sin^2\phi} - \cos\phi\right] = -\frac{8}{3}\sqrt{\frac{2}{\pi}}\, S, \qquad (3\text{-}41a)$$

$$\lim_{S\to\infty} C_{R1}(S) = +\frac{4\sqrt{2}}{3\pi^2} S \int_0^\infty dw'_{1z} w'^2_{1z} \int_0^\infty dw_{R1} \int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta_\sigma \sin\theta_\sigma |\cos^3\theta_\sigma| \int_0^{2\pi} d\phi_\sigma$$

$$\exp\left[-w'^2_{1z}\{w^2_{R1} + 2w_{R1}\mathrm{tg}\theta_\sigma \cos(\phi-\phi_\sigma) + \mathrm{tg}^2\theta_\sigma + 1\}\right]\left[\frac{1-|\cos^3\phi|}{\sin^2\phi} - \cos\phi\right] = \frac{16}{15}\sqrt{\frac{2}{\pi}}\, S.$$

(3-41b)

The total value of the coefficient C_1 is then obtained from (3-36). We thus conclude that in the limit of high speed ratios

$$\lim_{S\to\infty} C_{H1}(S) = -\frac{8}{3}\sqrt{\frac{2}{\pi}}\, S \simeq -2.128\ S \qquad (3\text{-}42a)$$

$$\lim_{S\to\infty} C_{R1}(S) = +\frac{16}{15}\sqrt{\frac{2}{\pi}}\, S \simeq +0.851\ S \qquad (3\text{-}42b)$$

$$\lim_{S\to\infty} C_1(S) = \lim_{S\to\infty}\{C_{H1}(S) + 2C_{R1}(S)\} = -\frac{8}{15}\sqrt{\frac{2}{\pi}}\, S \simeq -0.426\ S \qquad (3\text{-}42c)$$

in agreement with the result earlier obtained by Willis [20,21].

3.5 Drag Coefficient C_1 of a Disc at Low Speed Ratios

At low speed ratios we can expand the Maxwell distribution (3-6) in terms of a Taylor series around S=0. If we neglect terms of higher order in S, then

$$f^*(\vec{w}_1;\vec{S})f^*(\vec{w}_2;\vec{S}) = \frac{1}{\pi^3} e^{-(w_1^2 + w_2^2)}\{1-2S(w_{1z} + w_{2z})\}, \qquad (3\text{-}43)$$

to be substituted into the collision integrals (3-33). The zeroth order term in (3-43) does not contribute to the drag coefficient as a result of (3-34). We also note that we can integrate over $\vec{w}_1$ in the collision integrals for C_{H1} and C_{R1}. In the limit of low velocity we thus obtain

from (3-33) and (3-34)

$$\lim_{S\to 0} C_{H1}(S) = -\frac{8\sqrt{2}}{3\pi^3 S}\int d\vec{w}_2 e^{-w_2^2}\int d\vec{w}_1' e^{-w_1'^2} w_{12}'\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right)^2 I(\vec{w}_{H1})\,\Theta(-w_{2z})\,\Theta(w_{1z}'), \tag{3-44a}$$

$$\lim_{S\to 0} C_{H2}(S) = +\frac{8\sqrt{2}}{3\pi^3 S}\int d\vec{w}_1 e^{-w_1^2}\int d\vec{w}_2 e^{-w_2^2}(-w_{1z}-w_{2z})w_{12}\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H2})$$
$$\cdot\Theta(w_{1z})\,\Theta(-w_{2z}) \quad , \tag{3-44b}$$

$$\lim_{S\to 0} C_{R1}(S) = +\frac{8\sqrt{2}}{3\pi^3 S}\int d\vec{w}_2 e^{-w_2^2}\int d\vec{w}_1' e^{-w_1'^2}\frac{1}{\pi}\int d\hat{\sigma}_{12}\,|\vec{w}_{12}'\cdot\hat{\sigma}_{12}|$$
$$\cdot\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right)\left(\frac{\sqrt{\pi}}{2} - w_{1z}''\right) I(\vec{w}_{R1})\,\Theta(w_{1z}')\,\Theta(-\vec{w}_{12}'\cdot\hat{\sigma}_{12})\,\Theta(-w_{1z}''), \tag{3-44c}$$

$$\lim_{S\to 0} C_{R2}(S) = -\frac{8\sqrt{2}}{3\pi^3 S}\int d\vec{w}_1 e^{-w_1^2}\int d\vec{w}_2 e^{-w_2^2}\frac{1}{\pi}\int d\hat{\sigma}_{12}\,|\vec{w}_{12}\cdot\hat{\sigma}_{12}|$$
$$\cdot(-w_{1z}-w_{2z})\left(\frac{\sqrt{\pi}}{2} - w_{1z}''\right) I(\vec{w}_{R2})\,\Theta(w_{1z})\,\Theta(-\vec{w}_{12}\cdot\hat{\sigma}_{12})\,\Theta(-w_{1z}''), \tag{3-44d}$$

We recall that we work in a coordinate system with the Z-axis in the direction of $-\vec{S}$ and the X-axis in the plane through $\vec{S}$ and $\vec{r}_1$. The polar and arimuthal angles of any vector $\vec{a}$ in this coordinate system are indicated by θ_a and ϕ_a. We express the perihelion vector $\hat{\sigma}_{12}$ in terms of a polar angle θ_σ' and an arimuthal angle ϕ_σ' in an auxiliary coordinate system X'Y'Z' with the Z'-axis in the direction of $\vec{w}_{12}'$ ($=\vec{w}_{12}$ in (3-44d)) and with the X'-axis in the plane through $\vec{S}$ and $\vec{w}_{12}'$. The Cartesian components of any vector $\vec{a}$ in the auxiliary system are related to its components in the original XYZ system by

$$\begin{pmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{pmatrix} = R_y(\theta_{w12'})R_z(\phi_{w12'})\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \tag{3-45}$$

Here $\theta_{w12'}$ and $\phi_{w12'}$ are the polar coordinates of $\vec{w}_{12}'$ in the XYZ system and R_y and R_z are the same rotation matrices as those used in earlier reports [31,33].

We conclude that the drag coefficient C_1 in the limit of low velocity can be written as

$$\lim_{S\to o} C_1(S) = \left[\lim_{S\to o} C_{H1}(S) + C_{H2}(S) + 2\{C_{R1}(S) + C_{R2}(S)\}\right] , \tag{3-46}$$

with

$$\lim_{S\to o} C_{H1}(S) = -\frac{8\sqrt{2}}{3\pi^3 S}\int_o^\infty dw_2 w_2{}^2 e^{-w_2{}^2}\int_o^\infty dw_1' w_1'{}^2 e^{-w_1'{}^2}\int_{\pi/2}^{\pi} d\theta_{w2}\sin\theta_{w2}\int_o^{2\pi} d\phi_{w2}$$

$$\cdot\int_o^{\pi/2} d\theta_{w1'}\sin\theta_{w1'}\int_o^{2\pi} d\phi_{w1'} w_{12}'\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right)^2 I(\vec{w}_{H1}) , \tag{3-46a}$$

$$\lim_{S\to o} C_{H2}(S) = +\frac{8\sqrt{2}}{3\pi^3 S}\int_o^\infty dw_2 w_2{}^2 e^{-w_2{}^2}\int_o^\infty dw_1 w_1{}^2 e^{-w_1{}^2}\int_{\pi/2}^{\pi} d\theta_{w2}\sin\theta_{w2}\int_o^{2\pi} d\phi_{w2}$$

$$\cdot\int_o^{\pi/2} d\theta_{w1}\sin\theta_{w1}\int_o^{2\pi} d\phi_{w1}(-w_{1z}-w_{2z})w_{12}\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right)I(\vec{w}_{H2}) , \tag{3-46b}$$

$$\lim_{S\to o} C_{R1}(S) = +\frac{8\sqrt{2}}{3\pi^3 S}\int_o^\infty dw_2 w_2{}^2 e^{-w_2{}^2}\int_o^\infty dw_1' w_1'{}^2 e^{-w_1'{}^2}\int_o^{\pi} d\theta_{w2}\sin\theta_{w2}\int_o^{2\pi} d\phi_{w2}$$

$$\cdot\int_o^{\pi/2} d\theta_{w1'}\sin\theta_{w1'}\int_o^{2\pi} d\phi_{w1'}\ \frac{1}{\pi}\int_{\pi/2}^{\pi} d\theta_\sigma'\sin\theta_\sigma'|\cos\theta_\sigma'|\int_o^{2\pi} d\phi_\sigma'$$

$$\cdot\ w_{12}'\left(\frac{\sqrt{\pi}}{2} - w_{2z}\right)\left(\frac{\sqrt{\pi}}{2} - w_{1z}''\right) I(\vec{w}_{R1})\Theta(-w_{1z}'') , \tag{3-46c}$$

$$\lim_{S\to o} C_{R2}(S) = -\frac{8\sqrt{2}}{3\pi^3 S}\int_o^\infty dw_2 w_2{}^2 e^{-w_2{}^2}\int_o^\infty dw_1 w_1{}^2 e^{-w_1{}^2}\int_o^{\pi} d\theta_{w2}\sin\theta_{w2}\int_o^{2\pi} d\phi_{w2}$$

$$\int_o^{\pi/2} d\theta_{w1}\sin\theta_{w1}\int_o^{2\pi} d\phi_{w1}\ \frac{1}{\pi}\int_{\pi/2}^{\pi} d\theta_\sigma'\sin\theta_\sigma'|\cos\theta_\sigma'|\int_o^{2\pi} d\phi_\sigma'(-w_{1z}-w_{2z})w_{12}\left(\frac{\sqrt{\pi}}{2} - w_{1z}''\right)I(\vec{w}_{R2})\Theta(-w_{1z}'')$$

(3-46d)

TABLE II

Collision integrals for the drag coefficient C_1 of a disc in the low velocity limit.

$$\begin{cases} \lim_{S\to 0} C_{H1}(S) = -(3.94 \pm 0.13)S^{-1} \\ \lim_{S\to 0} C_{H2}(S) = +(0.23 \pm 0.03)S^{-1} \\ \lim_{S\to 0} C_{H}(S) = \lim_{S\to 0} \{C_{H1}(S) + C_{H2}(S)\} = -(3.71 \pm 0.10)S^{-1} \end{cases}$$

$$\begin{cases} \lim_{S\to 0} C_{R1}(S) = +(1.14 \pm 0.07)S^{-1} \\ \lim_{S\to 0} C_{R2}(S) = -(0.05 \pm 0.02)S^{-1} \\ \lim_{S\to 0} C_{R}(S) = \lim_{S\to 0} \{C_{R1}(S) + C_{R2}(S)\} = +(1.10 \pm 0.08)S^{-1} \end{cases}$$

$$\lim_{S\to 0} C_{1}(S) = \lim_{S\to 0} \{C_{H}(S) + 2C_{R}(S)\} = -(1.51 \pm 0.08)S^{-1}$$

These collision integrals were evaluated numerically by a Monte Carlo method. That is, the integrals were estimated by averaging the integrand over a set of N random points selected in the integration region according to a suitable predetermined probability density function [34]. For this purpose we employed the same method and subroutines that were previously used by Gillespie and Sengers in a calculation of three-particle collision integrals for a gas of hard spheres and reported in AEDC-TR-73-171 [31]. The numerical results, together with their estimated standard deviations, obtained by averaging the integrand over 50,000 points, are presented in Table II.

3.6 Drag Coefficient C_1 of a Disc at Arbitrary Speed Ratios

For arbitrary values of the speed ratio S we need to substitute the complete Maxwell distribution (3-6) into the collision integrals (3-33). Thus

$$C_{H1}^{+}(S) = -\frac{2\sqrt{2}}{3\pi^3 S^2} J(S) \int d\vec{w}_2 e^{-(\vec{w}_2-\vec{S})^2} \int d\vec{w}_1' e^{-w_1'^2} w_{12}' \left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H1}) \cdot \Theta(-w_{2z})\, \Theta(w_{1z}') , \quad (3\text{-}47a)$$

$$C_{H2}^{+}(S) = +\frac{2\sqrt{2}}{3\pi^3 S^2} \int d\vec{w}_1 e^{-(\vec{w}_1-\vec{S})^2} \int d\vec{w}_2 e^{-(\vec{w}_2-\vec{S})^2} w_{12} \left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H2}) \cdot \Theta(w_{1z})\, \Theta(-w_{2z}) , \quad (3\text{-}47b)$$

$$C_{R1}^{+}(S) = +\frac{2\sqrt{2}}{3\pi^3 S^2} J(S) \int d\vec{w}_2 e^{-(\vec{w}_2-\vec{S})^2} \int d\vec{w}_1' e^{-w_1'^2} \frac{1}{\pi} \int d\hat{\sigma}_{12} \left|\vec{w}_{12}'\cdot\hat{\sigma}_{12}\right| \cdot \left(\frac{\sqrt{\pi}}{2} - w_{1z}''\right) I(\vec{w}_{R1})\, \Theta(w_{1z}')\, \Theta(-\vec{w}_{12}'\cdot\hat{\sigma}_{12})\, \Theta(-w_{1z}'') , \quad (3\text{-}47c)$$

$$C_{R2}^{+}(S) = -\frac{2\sqrt{2}}{3\pi^3 S^2} \int d\vec{w}_1 e^{-(\vec{w}_1-\vec{S})^2} \int d\vec{w}_2 e^{-(\vec{w}_2-\vec{S})^2} \frac{1}{\pi} \int d\hat{\sigma}_{12} \left|\vec{w}_{12}\cdot\hat{\sigma}_{12}\right| \cdot \left(\frac{\sqrt{\pi}}{2} - w_{1z}''\right) I(\vec{w}_{R2})\, \Theta(w_{1z})\, \Theta(-\vec{w}_{12}\cdot\hat{\sigma}_{12})\, \Theta(-w_{1z}'') , \quad (3\text{-}47d)$$

where

$$J(S) \equiv \frac{2}{\pi} \int d\vec{w}_1 e^{-(\vec{w}_1-\vec{S})^2} |w_{1z}|\, \Theta(-w_{1z}) = e^{-S^2} + \pi^{1/2} S(1+\mathrm{erf}\ S) . \quad (3\text{-}48)$$

We can express these integrals in terms of the same variables introduced in the preceding section and obtain

$$C_{H1}^{+}(S) = -\frac{2\sqrt{2}}{3\pi^3 S^2} J(S) \int_0^\infty dw_2 w_2^2 e^{-w_2^2} \int_0^\infty dw_1' w_1'^2 e^{-w_1'^2} \int_{\pi/2}^{\pi} d\theta_{w2} \sin\theta_{w2} \int_0^{2\pi} d\phi_{w2} \cdot \int_0^{\pi/2} d\theta_{w1'} \sin\theta_{w1'} \int_0^{2\pi} d\phi_{w1'} e^{-S(S+2w_{2z})} w_{12}' \left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H1}), \quad (3\text{-}49a)$$

$$C^{+}_{H2}(S) = + \frac{2\sqrt{2}}{3\pi^3 S^2} \int_0^\infty dw_2 w_2{}^2 e^{-w_2{}^2} \int_0^\infty dw_1 w_1{}^2 e^{-w_1{}^2} \int_{\pi/2}^{\pi} d\theta_{w2} \sin\theta_{w2} \int_0^{2\pi} d\phi_{w2}$$

$$\cdot \int_0^{\pi/2} d\theta_{w1} \sin\theta_{w1} \int_0^{2\pi} d\phi_{w1} e^{-2S(S+w_{1z}+w_{2z})} w_{12} \left(\frac{\sqrt{\pi}}{2} - w_{2z}\right) I(\vec{w}_{H2}) \ , \qquad (3\text{-}49b)$$

$$C^{+}_{R1}(S) = + \frac{2\sqrt{2}}{3\pi^3 S^2} J(S) \int_0^\infty dw_2 w_2{}^2 e^{-w_2{}^2} \int_0^\infty dw'_1 w'_1{}^2 e^{-w'_1{}^2} \int_0^{\pi} d\theta_{w2} \sin\theta_{w2} \int_0^{2\pi} d\phi_{w2}$$

$$\cdot \int_0^{\pi/2} d\theta_{w1'} \sin\theta_{w1'} \int_0^{2\pi} d\phi_{w1'} \frac{1}{\pi} \int_{\pi/2}^{\pi} d\theta'_\sigma \sin\theta'_\sigma |\cos\theta'_\sigma| \int_0^{2\pi} d\phi'_\sigma e^{-S(S+w_{2z})} w'_{12} \left(\frac{\sqrt{\pi}}{2} - w''_{1z}\right) I(\vec{w}_{R1}) \Theta(-w''_{1z})$$

(3-49c)

$$C^{+}_{R2}(S) = - \frac{2\sqrt{2}}{3\pi^3 S^2} \int_0^\infty dw_2 w_2{}^2 e^{-w_2{}^2} \int_0^\infty dw_1 w_1{}^2 e^{-w_1{}^2} \int_0^{\pi} d\theta_{w2} \sin\theta_{w2} \int_0^{2\pi} d\phi_{w2}$$

$$\cdot \int_0^{\pi/2} d\theta_{w1} \sin\theta_{w1} \int_0^{2\pi} d\phi_{w1} \frac{1}{\pi} \int_{\pi/2}^{\pi} d\theta'_\sigma \sin\theta'_\sigma |\cos\theta'_\sigma| \int_0^{2\pi} d\phi'_\sigma e^{-2S(S+w_{1z}+w_{2z})} w_{12} \left(\frac{\sqrt{\pi}}{2} - w''_{1z}\right) I(\vec{w}_{R2}) \Theta(-w''_{1z})$$

(3-49d)

For various values of the speed ratio S the collision integrals were again evaluated numerically.† The results obtained by averaging the integrand over 40,000 random points are presented in Tables III and IV. From the figures presented in Table III we note that the contributions C_{H2} and C_{R2} are small; they decrease rapidly with increasing values of the speed ratio S and can be neglected for $S \geq 2$. The results for $S = 0.1$ are in satisfactory agreement with the low velocity values earlier presented in Table II, while at high velocities the results do indeed approach the exact values given in (3-42).

†Further details concerning the procedures used in calculating the collision integrals are documented in a Ph.D. thesis to be submitted by Y. Y. Lin Wang.

TABLE III

Collision integrals for the drag coefficient C_1 of a disc ($S \leq 2$)

S	$C_{H1} \times S$	$C_{H2} \times S$	$C_{R1} \times S$	$C_{R2} \times S$	$C_1 \times S$
0.1	-3.89±0.11	+0.21±0.02	+1.16±0.09	-0.04±0.03	-1.44±0.20
0.5	-4.96±0.15	+0.16±0.02	+1.42±0.11	-0.03±0.02	-2.02±0.25
1.0	-8.01±0.16	+0.068±0.005	+2.16±0.10	-0.014±0.007	-3.65±0.22
1.5	-12.3±0.7	+0.015±0.002	+2.7±0.2	-0.0019±0.0028	-6.8±0.7
2.0	-15.0±1.4	+0.0019±0.0005	+3.8±0.6	-0.0005±0.0003	-7.4±1.6

TABLE IV

Collision integrals for the drag coefficient C_1 of a disc ($S \geq 2$)

S	C_{H1}/S	C_{R1}/S	C_1/S
2.0	-4.62±0.04	+1.37±0.02	-1.87±0.05
3.5	-3.34±0.02	+1.12±0.02	-1.09±0.04
5.0	-2.99±0.02	+1.04±0.02	-0.90±0.04
7.5	-2.68±0.01	+0.96±0.01	-0.76±0.02
10	-2.53±0.01	+0.92±0.01	-0.69±0.02
15	-2.42±0.02	+0.90±0.01	-0.62±0.02
20	-2.32±0.01	+0.89±0.01	-0.54±0.02
30	-2.26±0.01	+0.88±0.01	-0.54±0.03
40	-2.22±0.01	+0.87±0.02	-0.48±0.03
50	-2.19±0.01	+0.86±0.01	-0.46±0.03

3.7 Discussion of Results for the Drag of a Disc.

In summarizing our results for the drag in the nearly free molecular regime, we write the drag coefficient C_D as

$$C_D = C_o\left[1+\frac{C_1}{C_o}K^{-1}\right] = C_o\left[1+\left(\frac{C_H+2C_R}{C_o}\right)K^{-1}\right], \tag{3-51}$$

with the inverse Knudsen number K^{-1} defined in (3-10). The term $(C_1/C_o)K^{-1}$ represents the first inverse Knudsen number correction to the drag force relative to the magnitude of the drag force in the free molecular flow limit. The contribution $C_H = C_{H1}+C_{H2}$ accounts for the effect of the hypothetical collisions and the contribution $C_R = C_{R1} + C_{R2}$ for the effect of the recollisions and cyclic collisions.

We have calculated the coefficients C_o and C_1 over the entire range of speed ratios between zero and infinity, assuming that the temperature of the reflected molecules is the same as the temperature of the molecules in the incident gas stream. The free molecular flow drag coefficient C_o is given by (3-20), while the coefficient C_1 of the first inverse Knudsen correction was evaluated numerically. The results are summarized in Table V and plotted graphically as a function of the speed ratio S in Figs. 10 and 11.

In the nearly free molecular flow regime the drag coefficient decreases with decreasing Knudsen number. The sign of the effect is determined by the contribution $C_H \approx C_{H1}$, i.e. by the fact that the reflected molecules prevent incident molecules from striking the object.

The dependence of the drag coefficient on the speed ratio is quite different whether the speed ratio is smaller or larger than unity. At low velocities the drag force is proportional to the speed ratio S. As a

TABLE V

Drag coefficient of a disc in the nearly free molecular flow regime as a function of the speed ratio S.

$$C_D = C_o\left[1+\frac{C_1}{C_o}K^{-1}\right] = C_o\left[1+\left(\frac{C_H+2C_R}{C_o}\right)K^{-1}\right];\ K^{-1} = \sqrt{2}\pi n\sigma^2 R$$

S	C_o	C_H/C_o	C_R/C_o	C_1/C_o
0	4.03 S^{-1}	−0.92±0.02	+0.27±0.02	−0.38±0.02
0.1	4.04 S^{-1}	−0.91±0.02	+0.28±0.02	−0.36±0.05
0.5	4.21 S^{-1}	−1.14±0.03	+0.33±0.03	−0.48±0.06
1.0	4.72 S^{-1}	−1.68±0.03	+0.45±0.02	−0.77±0.05
1.0	4.72	−(1.68±0.03)S	+(0.45±0.02)S	−(0.77±0.05)S
2.0	3.14	−(1.47±0.01)S	+(0.44±0.01)S	−(0.60±0.02)S
3.5	2.59	−(1.29±0.01)S	+(0.43±0.01)S	−(0.43±0.01)S
5.0	2.39	−(1.25±0.01)S	+(0.43±0.01)S	−(0.38±0.01)S
7.5	2.25	−(1.19±0.01)S	+(0.43±0.01)S	−(0.34±0.01)S
10	2.19	−(1.16±0.01)S	+(0.42±0.01)S	−(0.32±0.01)S
15	2.12	−(1.14±0.01)S	+(0.42±0.01)S	−(0.29±0.01)S
20	2.09	−(1.11±0.01)S	+(0.43±0.01)S	−(0.26±0.01)S
30	2.06	−(1.10±0.01)S	+(0.43±0.01)S	−(0.26±0.01)S
40	2.04	−(1.09±0.01)S	+(0.43±0.01)S	−(0.23±0.01)S
50	2.04	−(1.08±0.01)S	+(0.42±0.01)S	−(0.23±0.01)S
∞	2.00	−1.064S	+0.426S	−0.213S

result at small velocities both C_o and C_1 become inverse proportional to the speed ratio S and the ratio C_1/C_o becomes independent of S. On the other hand, at large velocities the free molecular drag force varies with the square of the stream velocity, while the first inverse Knudsen correction to the drag force varies with the third power of the stream velocity. Thus the ratio C_1/C_o becomes proportional to the speed ratio S; this feature is encountered for a large class of" objects and will be further discussed in Section 4.7. In the limit of infinite speed ratio our result agrees with the value calculated by Willis et al. [21] for the drag coefficient of a disc in a beam of hard spherical molecules.

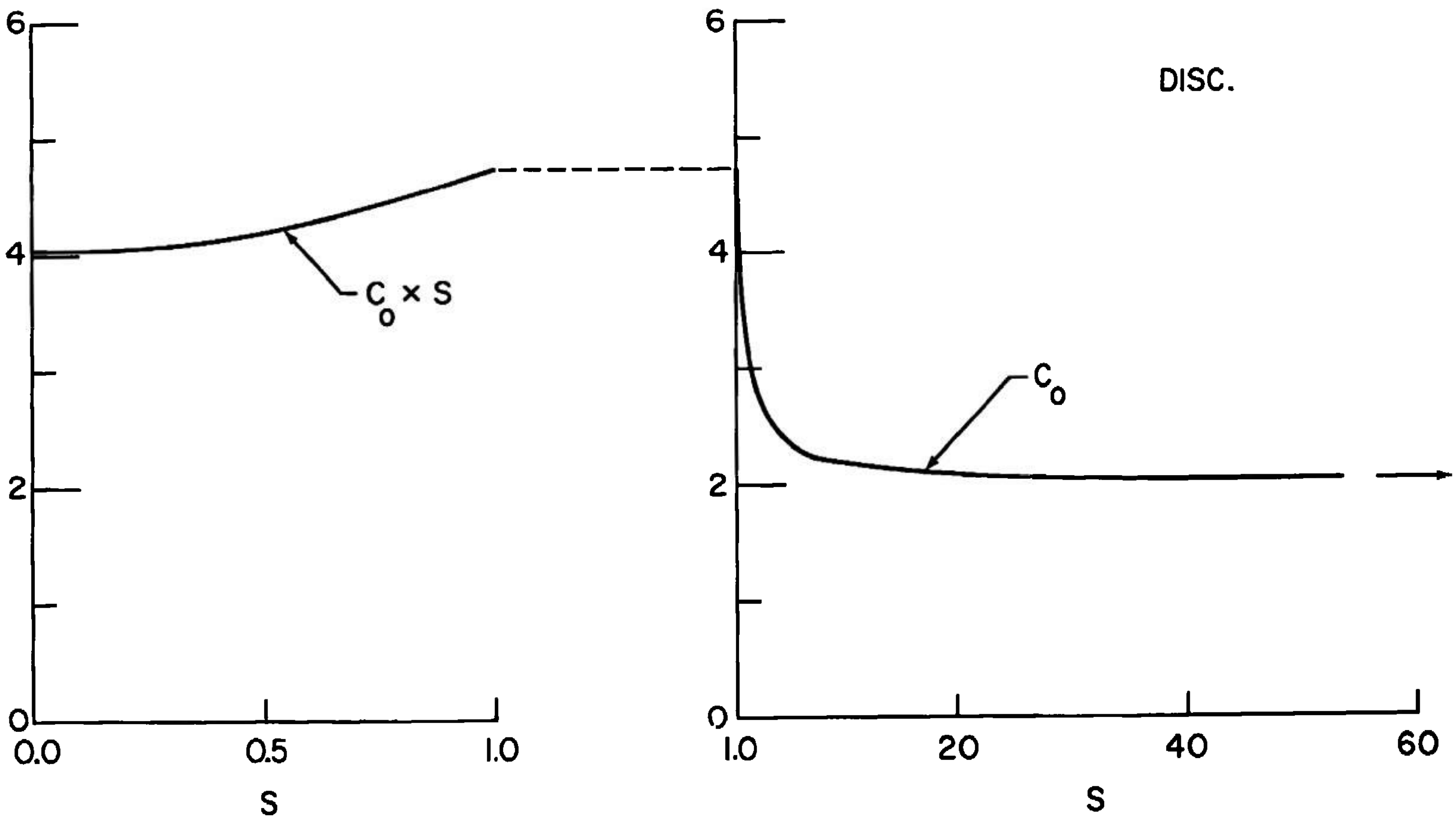


Figure 10. The drag coefficient C_o of a disc in the free molecular flow regime as a function of the speed ratio S.

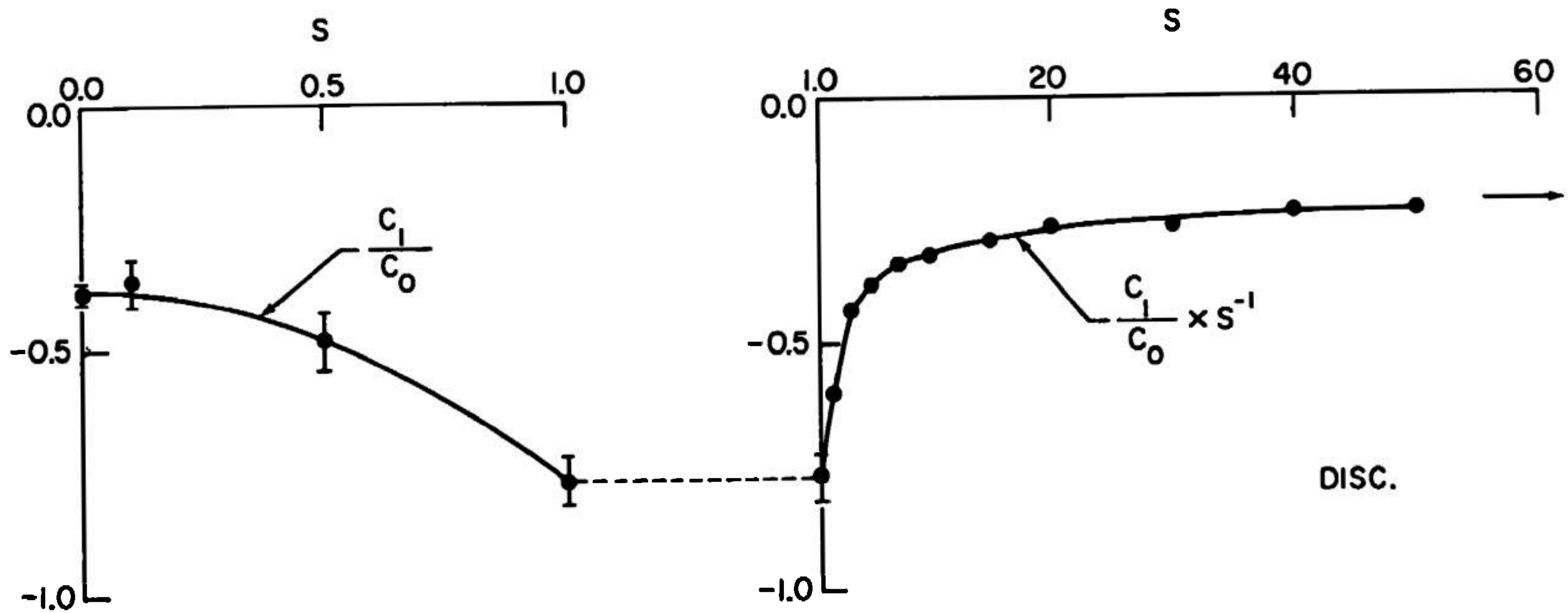


Figure 11. The coefficient C_1/C_0 of the first inverse Knudsen number correction to the drag coefficient of a disc as a function of the speed ratio S.

CHAPTER IV

DRAG COEFFICIENT OF A SPHERE IN THE NEARLY FREE MOLECULAR FLOW REGIME

4.1 Introduction

As a second application of our method we consider the drag force exerted on a sphere. A number of investigators have attempted to calculate the drag coefficient of a sphere using various approximation methods. However, to our knowledge a solution based on the full nonlinear Boltzmann equation that covers the entire range of speed ratios has not been presented earlier.

We use the same dimensionless quantities that were introduced in Section 3.1 when discussing the drag coefficient of a disc. The drag coefficient in the nearly free molecular flow regime is again represented by (3-11)

$$C_D = C_o + C_o K^{-1} , \tag{4-1}$$

with

$$K^{-1} = \sqrt{2}\, \pi n \sigma^2 R \quad , \tag{4-2}$$

where R now refers to the radius of the sphere. The drag coefficient C_o in the free molecular flow limit $K^{-1} \to o$ is in analogy to (3-12)

$$C_o = - \frac{2}{\pi S^2} \int d\vec{w}\ f^*(\vec{w};\vec{S}) \int_{\vec{w}\cdot\hat{R}<o} d\hat{R}\ |\vec{w}\cdot\hat{R}| \int d\vec{w}'\, \eta^*(\vec{w}')\, (\vec{w}'-\vec{w}) . \hat{S} . \tag{4-3}$$

The coefficient C_1 of the first inverse Knudsen number correction

$$C_1 = C_{H1} + C_{H2} + 2(C_{R1} + C_{R2}) \quad , \tag{4-4}$$

is again determined by the collision integrals listed in Table I, if $\vec{r}_1$ and $\hat{n}_1$ are both identified with the unit vector $\hat{R}_1$ in the direction of $\vec{R}_1$ and $\vec{r}_2$ and $\hat{n}_2$ with the unit vector $\hat{R}_2$ in the direction of $\vec{R}_2$ and where $\vec{R}_1$ and $\vec{R}_2$ are indicated in Figs. 4 and 6. The dimensionless velocities $\vec{w}_i$, $\vec{w}_i'$, $\vec{w}_i''$ and $\vec{w}_i'''$ may be identified with the velocities $\vec{v}_i$, $\vec{v}_i'$, $\vec{v}_i''$ and $\vec{v}_i'''$ shown in Figs. 4 and 6, provided that the radius of the sphere is normalized to unity.

4.2 Drag Coefficient C_o of a Sphere in the Free Molecular Flow Regime.

The drag coefficient C_o of a sphere in the free molecular flow regime is given by (4-3). Let θ_R be the polar angle of $\hat{R}$ with respect to $-\vec{S}$; because of symmetry we can integrate over the azimuthal angle of $\hat{R}$. Let θ_w' be the polar angle of $\vec{w}$ with respect to $\hat{R}$ and ϕ_w' the azimuthal angle of $\vec{w}$ with the plane through $\vec{S}$ and $\hat{R}$ as initial plane. Then, using (3-14),

$$C_o(S) = \frac{4}{\pi^{3/2} S^2} \int_0^{\pi} d\theta_R \sin\theta_R \int_0^{\infty} dw\, w^3 \int_{\pi/2}^{\pi} d\theta_w' \sin\theta_w' \, |\cos\theta_w'| \int_0^{2\pi} d\phi_w'$$

$$\cdot\, e^{-(w^2+S^2)+2S(\vec{w}\cdot\hat{S})} \left\{ (\vec{w}\cdot\hat{S}) + \frac{\sqrt{\pi}}{2}\cos\theta_R \right\} , \tag{4-5}$$

with

$$\vec{w}\cdot\hat{S} = w(\sin\theta_R \sin\theta_w' \cos\phi_w' - \cos\theta_R \cos\theta_w') \quad . \tag{4-5a}$$

The integral can be evaluated analytically with the result [1,10,32]

$$\boxed{C_o(S) = \frac{(2S^2+1)}{\sqrt{\pi}\, S^3} e^{-S^2} + \frac{(4S^4+4S^2-1)}{2S^4} \operatorname{erf} S + \frac{2\sqrt{\pi}}{3S}} \quad . \tag{4-6}$$

In the low and high speed limits the drag coefficient $C_o(S)$ reduces to

$$\lim_{S\to 0} C_o(S) = \frac{2\pi+16}{3\sqrt{\pi}\, S} \simeq \frac{4.191}{S} \tag{4-7}$$

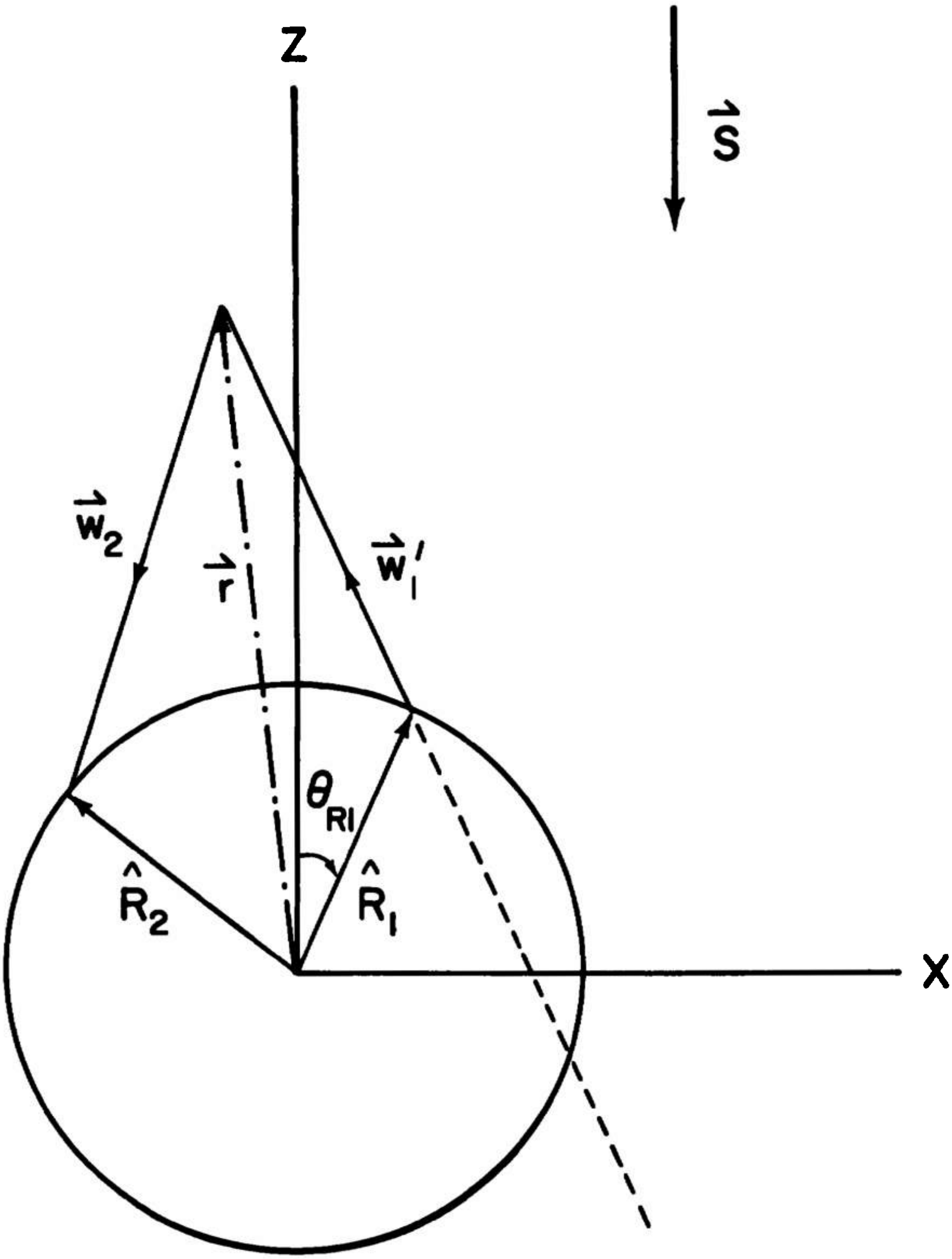


Figure 12. Geometry of an hypothetical collision for a sphere.

and

$$\lim_{S\to\infty} C_o(S) = 2. \tag{4-8}$$

4.3 Reduction of Collision Integrals for C_1†

The coefficient C_1 of the first inverse Knudsen number correction to the drag coefficient is again determined by the collision integrals in Table I with $\vec{r}_1 = \hat{n}_1 = \hat{R}_1$ and $\vec{r}_2 = \hat{n}_2 = \hat{R}_2$. The fact that each collision integral is related to a particular collision sequence is accounted for by the δ-functions. In this Section we integrate over the δ-functions and formulate explicitly the conditions for the occurrance of these collision sequences.

As an example we consider the collision integrals associated with the hypothetical collisions (cf. Fig. 6). The geometry of an H1-sequence is indicated in Fig. 12. The Z-axis is taken in the direction of $-\vec{S}$ and the X-axis in the plane through $\vec{S}$ and $\hat{R}_1$. Molecule 1 is emitted from the surface at $\hat{R}_1$ with velocity $\vec{w}_1'$. Molecule 2 with incident velocity $\vec{w}_2$ collides with molecule 1 after a time τ_1^*. We introduce the vector $\vec{r}$ defined as

$$\vec{r} \equiv \hat{R}_1 + \vec{w}_1' \tau_1^* \tag{4-9}$$

which determines the location of the collision between molecules 1 and 2 relative to the center of the sphere. Molecule 2 continues to proceed with its initial velocity $\vec{w}_2$ and strikes the surface of the object after a time τ_2^*. Although not indicated explicitly in Fig. 12, the vectors $\vec{r}$ and $\vec{R}_2$ are not restricted to the XZ-plane.

The δ-function in the expression (I-1) for C_{H1} can be rewritten in terms of the velocity $\vec{w}_2$ and the distance vector $\vec{r}$

†Sections 4.3 and 4.4 were prepared in collaboration with Dr. W. A. Kuperman.

$$|\vec{w}_2\cdot\hat{R}_2|\ \delta^3(\hat{R}_1-\hat{R}_2+\vec{w}_1'\tau_1^*+\vec{w}_2\tau_2^*)= \tag{4-10}$$

$$=\delta^3\left(\tau_2^*+\frac{r^2-\hat{R}_2\cdot\vec{r}}{\vec{w}_2\cdot\vec{r}}\right)\delta^2(\hat{R}_2-\vec{r}+\hat{w}_2\{\hat{w}_2\cdot\vec{r}+\sqrt{1-(\hat{w}_2\times\vec{r})^2}\})\Theta(-\hat{w}_2\cdot\vec{r})\Theta(1-\{\hat{w}_2\times\vec{r}\}^2).$$

The conditions imposed by the Heaviside functions in (4-10) have an obvious geometrical meaning. The condition $\hat{w}_2\cdot\vec{r} < 0$ guarantees that molecule 2 approaches the sphere and the condition $(\hat{w}_2\times\vec{r})^2 < 1$ ensures that the impact parameter is smaller than unity.

The conditions for the H2-sequence (cf. Fig. 6b) can be related to the conditions for the H1-sequence, if we identify the velocity $\vec{w}_1'$ in (4-9) and (4-10) with the initial velocity $\vec{w}_1$ and if we register the first collision not at the position where molecule 1 *enters* the sphere, but at the position where molecule 1 *leaves* the sphere.

Hence, if we integrate over τ_2^*, $\hat{R}_2$ and $\hat{\sigma}_{12}$(cf.(3-32)) and use (3-14), the collision integrals (I-1) and (I-2) reduce to

$$C_{H1}(S)=-\frac{\sqrt{2}}{\pi S^2}\int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S})f^*(\vec{w}_2;\vec{S})\int_0^\infty d\tau_1^*\int_{\vec{w}_1\cdot\hat{R}_1<0} d\hat{R}_1\ |\vec{w}_1\cdot\hat{R}_1|$$

$$\cdot\int d\vec{w}_1'\eta^*(\vec{w}_1')w_{12}'(\vec{w}_2-\frac{\sqrt{\pi}}{2}\hat{R}_2).\hat{S}\ \Theta(-\hat{w}_2\cdot\vec{r})\Theta(1-\{\hat{w}_2\times\vec{r}\}^2)\quad , \tag{4-11a}$$

$$C_{H2}(S)=+\frac{\sqrt{2}}{\pi S^2}\int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S})f^*(\vec{w}_2;\vec{S})\int_0^\infty d\tau_1^*\int_{\vec{w}_1\cdot\hat{R}_1>0} d\hat{R}_1\ |\vec{w}_1\cdot\hat{R}_1|$$

$$\cdot w_{12}(\vec{w}_2-\frac{\sqrt{\pi}}{2}\hat{R}_2).\hat{S}\ \Theta(-\hat{w}_2\cdot\vec{r})\Theta(1-\{\hat{w}_2\times\vec{r}\}^2), \tag{4-11b}$$

with

$$\hat{R}_2=\vec{r}-\hat{w}_2\left\{\hat{w}_2\cdot\vec{r}+\sqrt{1-(\hat{w}_2\times\vec{r})^2}\right\}. \tag{4-11c}$$

The collision integrals (I-3) and (I-4) can be treated in the same manner, except that we cannot integrate analytically over the collision vector $\hat{\sigma}_{12}$. Thus

$$C_{R1}(S) = +\frac{\sqrt{2}}{\pi^2 S^2} \int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_{\vec{w}_1\cdot\hat{R}_1<0} d\hat{R}_1 \, |\vec{w}_1\cdot\hat{R}_1|$$

$$\cdot \int d\vec{w}_1' \eta^*(\vec{w}_1') \int_{\vec{w}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \, |\vec{w}_{12}'\cdot\hat{\sigma}_{12}| \, (\vec{w}_1'' - \frac{\sqrt{\pi}}{2}\hat{R}_2)\cdot\hat{S} \, \Theta(-\hat{w}_1''\cdot\vec{r})\Theta(1-\{\hat{w}_1''\times\vec{r}\}^2), \quad (4\text{-}12a)$$

$$C_{R2}(S) = -\frac{\sqrt{2}}{\pi^2 S^2} \int d\vec{w}_1 d\vec{w}_2 f^*(\vec{w}_1;\vec{S}) f^*(\vec{w}_2;\vec{S}) \int_0^\infty d\tau_1^* \int_{\vec{w}_1\cdot\hat{R}_1>0} d\hat{R}_1 \, |\vec{w}_1\cdot\hat{R}_1|$$

$$\cdot \int_{\vec{w}_{12}\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \, |\vec{w}_{12}\cdot\hat{\sigma}_{12}| \, (\vec{w}_1'' - \frac{\sqrt{\pi}}{2}\hat{R}_2)\cdot\hat{S} \, \Theta(-\hat{w}_1''\cdot\vec{r})\Theta(1-\{\hat{w}_1''\times\vec{r}\}^2), \quad (4\text{-}12b)$$

with

$$\hat{R}_2 = \vec{r} - \hat{w}_1''\left\{\hat{w}_1''\cdot\vec{r} + \sqrt{1-(\hat{w}_1''\times\vec{r})^2}\right\}. \quad (4\text{-}12c)$$

The velocity $\vec{w}_1''$ is again related to the integration variables by (3-35).

4.4 Drag Coefficient C_1 of a Sphere at High Speed Ratios.

In the high speed limit or (cold wall) beam limit

$$f^*(\vec{w}_i;\vec{S}) = \delta^3(\vec{w}_i - \vec{S}) \,, \quad (4\text{-}13)$$

and

$$\lim_{S\to\infty} C_1(S) = \lim_{S\to\infty} C_{H1}(S) + 2\lim_{S\to\infty} C_{R1}(S) \,, \quad (4\text{-}14)$$

as discussed in Section 3.4. Retaining only the leading term in S the collision integrals (4-11a) and (4-12a) reduce to

$$\lim_{S\to\infty} C_{H1}(S) = -\frac{\sqrt{2}}{\pi} S \int_0^\infty d\tau_1^* \int_{\hat{S}\cdot\hat{R}_1<0} d\hat{R}_1 \, |\hat{S}\cdot\hat{R}_1| \int d\vec{w}_1' \eta^*(\vec{w}_1') \Theta(-\hat{S}\cdot\vec{r})\Theta(1-\{\hat{S}\times\vec{r}\}^2), \quad (4\text{-}15)$$

$$\lim_{S\to\infty} C_{R1}(S) = +\frac{\sqrt{2}}{\pi^2} S \int_0^\infty d\tau_1^* \int_{\hat{S}\cdot\hat{R}_1<0} d\hat{R}_1 \, |\hat{S}\cdot\hat{R}_1| \int d\vec{w}_1' \eta^*(\vec{w}_1') \int_{\hat{S}\cdot\hat{\sigma}_{12}>0} d\hat{\sigma}_{12} (\hat{S}\cdot\hat{\sigma}_{12})^3$$

$$\cdot \Theta(-\hat{\sigma}_{12}\cdot\vec{r})\Theta(1-\{\hat{\sigma}_{12}\times\vec{r}\}^2) \,. \quad (4\text{-}16)$$

We express the vectors $\hat{R}_1(\theta_{R1},\phi_{R1})$, $\vec{r}(r,\theta_r,\phi_r)$ and $\vec{w}_1'(w_1',\theta_{w1'},\phi_{w1'})$ in terms of spherical polar coordinates in the coordinate system with the Z-axis in the direction of $-\vec{S}$. Because of symmetry we may integrate over the azimuthal angle ϕ_{R1} of $\hat{R}_1$ and take $\hat{R}_1$ in the XZ-plane as indicated in Fig. 12. Then

$$\lim_{S\to\infty} C_{H1}(S) = -\frac{2^{5/2}}{\pi} S \int_0^\infty d\tau_1^* \int_0^{\pi/2} d\theta_{R1} \sin\theta_{R1}\cos\theta_{R1} \int_0^\infty dw_1' w_1'^3 e^{-w_1'^2} \int_0^\pi d\theta_{w1'} \sin\theta_{w1'} \int_0^{2\pi} d\phi_{w1'}$$

$$\cdot(\hat{w}_1'\cdot\hat{R}_1)\Theta(\hat{w}_1'\cdot\hat{R}_1)\Theta(\cos\theta_r)\Theta(1-r\sin\theta_r)\ , \tag{4-17}$$

with

$$\hat{w}_1'\cdot\hat{R}_1 = \cos\theta_{R1}\cos\theta_{w1'} + \sin\theta_{R1}\sin\theta_{w1'}\cos\phi_{w1'}\ . \tag{4-17a}$$

The Heaviside functions in (4-17) impose the conditions

$$0<\theta_{R1}< \operatorname{arctg}(-\operatorname{cotg}\theta_{w1'}\sec\phi_{w1'})\ , \tag{4-18}$$

and

$$0.<\theta_r<\arcsin r^{-1}. \tag{4-19}$$

It follows from (4-9) that condition (4-19) is equivalent with

$$0<\tau_1^*< \frac{\sin\theta_{R1}}{w_1'\sin\theta_{w1'}}\left[-\cos\phi_{w1'}+\sqrt{\operatorname{cotg}^2\theta_{R1}+\cos^2\phi_{w1'}}\right]. \tag{4-20}$$

With the integration limits for τ_1^* and θ_{R1} determined by (4-20) and (4-18) the Heaviside functions in (4-17) are taken to be unity. All integrations can be performed analytically with the result

$$\lim_{S\to\infty} C_{H1}(S) = -S\sqrt{\frac{\pi}{2}}\ . \tag{4-21}$$

In order to evaluate the collision integral (4-16), as well as the collision integrals in the subsequent sections, we consider three different coordinate systems, to be referred to as the XYZ, X'Y'Z' and

X''Y''Z'' systems. The XYZ system is the original coordinate system with the Z-axis in the direction of $-\vec{S}$. The X'Y'Z' system has the Z'-axis in the direction of $\hat{R}_1$ and the X'-axis in the plane through $\hat{R}_1$ and $\hat{S}$. The X''Y''Z'' system has the Z''-axis in the direction of $\vec{r}$ and the X''-axis in the plane through $\vec{r}$ and $\vec{S}$. We express $\vec{w}_1'(w_1', \theta_{w1'}', \phi_{w1'}')$ in terms of polar coordinates in the X'Y'Z' system. The polar and azimuthal angle of $\hat{\sigma}_{12}$ in the original system are θ_σ, ϕ_σ and in the X''Y''Z'' system θ_σ'', ϕ_σ''. The Cartesian components of any vector $\vec{a}$ in the auxiliary systems are related to its components in the original XYZ system by

$$\begin{pmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{pmatrix} = R_y(\theta_{R1}) \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \quad \begin{pmatrix} a_{x''} \\ a_{y''} \\ a_{z''} \end{pmatrix} = R_y(\theta_r)\, R_z(\phi_r) \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}, \tag{4-22}$$

where R_y and R_z are the same rotation matrices as those used in earlier reports [31,33] and where θ_r and ϕ_r are the polar and azimuthal angle of the distance vector $\vec{r}$

$$\vec{r} = \hat{R}_1 + \vec{w}_1' \tau_1^* = \hat{R}_1 + \hat{w}_1' \left[-\cos\theta_{w1'}' + \sqrt{r^2 - \sin^2\theta_{w1'}'} \right]. \tag{4-23}$$

The collision integral (4-16) reads in terms of these variables

$$\lim_{S\to\infty} C_{R1}(S) = + \frac{4\sqrt{2}}{\pi^2} S \int_0^\infty d\tau_1^* \int_0^{\pi/2} d\theta_{R1} \sin\theta_{R1} \cos\theta_{R1} \int_0^\infty dw_1' w_1'^3 e^{-w_1'^2} \int_0^{\pi/2} d\theta_{w1'}' \sin\theta_{w1'}' \cos\theta_{w1'}'$$
$$\cdot \int_0^{2\pi} d\phi_{w1'}' \int_0^{\pi/2} d\theta_\sigma \sin\theta_\sigma \cos^3\theta_\sigma \int_0^{2\pi} d\phi_\sigma \Theta(-\cos\theta_\sigma'')\, \Theta(1 - r\sin\theta_\sigma''). \tag{4-24}$$

The Heaviside functions in (4-24) impose the condition

$$\pi - \arcsin r^{-1} < \theta_\sigma'' < \pi, \tag{4-25}$$

so that molecule 1 will indeed return to the sphere after the collision with molecule 2. We transform the integral over the *time* τ_1^* from 0 to ∞ into an integral over the *distance* r from 1 to ∞ [19]

$$d\tau_1^* = \frac{r\,dr}{w_1' \sqrt{r^2 - \sin^2\theta'_{w1'}}} \; . \qquad (4\text{-}26)$$

If we then integrate over w_1', we obtain

$$\lim_{S\to\infty} C_{R1}(S) = + \frac{\sqrt{2}}{\pi^{3/2}} S \int_0^{\pi/2} d\theta_{R1} \sin\theta_{R1} \cos\theta_{R1} \int_0^{\pi/2} d\theta'_{w1'} \sin\theta'_{w1'} \cos\theta'_{w1'} \int_0^{2\pi} d\phi'_{w1'}$$

$$\cdot \int_1^{\infty} dr \left(1 - r^{-2}\sin^2\theta'_{w1'}\right)^{-1/2} \int_0^{\pi/2} d\theta_\sigma \sin\theta_\sigma \cos^3\theta_\sigma \int_0^{2\pi} d\phi_\sigma \; \Theta(-\cos\theta''_\sigma)\,\Theta(1 - r\sin\theta''_\sigma) . \qquad (4\text{-}27)$$

The remaining integral was estimated numerically as a weighted average over 100,000 random points with the result $+(0.510 \pm 0.002)S$.

We thus conclude that in the limit of high speed ratios [16]

$$\lim_{S\to\infty} C_{H1}(S) = -\sqrt{\frac{\pi}{2}}\, S \equiv -1.253\, S \;, \qquad (4\text{-}28a)$$

$$\lim_{S\to\infty} C_{R1}(S) = +(0.510 \pm 0.002)S \;, \qquad (4\text{-}28b)$$

$$\lim_{S\to\infty} C_1(S) = \lim_{S\to\infty} \{C_{H1}(S) + 2C_{R1}(S)\} = -(0.233 \pm 0.004)S \, . \qquad (4\text{-}28c)$$

These results are in good agreement with the values $\lim_{S\to\infty} C_{H1}(S) = -(1.29 \pm 0.02)S$ and $\lim_{S\to\infty} C_{R1}(S) = +(0.511 \pm 0.002)S$ as calculated by Kuperman†.

---

† Equation (5-54) in the original thesis of Kuperman contains an error and should read $\hat{V}_{(a)} = R_x(-\tilde{\theta}) R_z(-\phi_{w1}) \hat{V}_{(k_1)}$. This change does not affect the value of C_1^{H1} quoted in Table 5.2 of his thesis, but the values quoted for C_1^{R1} and C_1^{C1} should be replaced with $C_1^{R1} = C_1^{C1} = (0.256 \pm 0.002)S/Kn'$ [35]. For the same reason the values quoted in Table 6.4 of Kuperman's thesis should be replaced with the values presented in Section 4.5 of this report.

4.5 Drag Coefficient C_1 of a Sphere at Low Speed Ratios.

At low speed ratios we can expand the Maxwell distributions in the collision integrals (4-11) and (4-12) in a Taylor series around S=0 and retain only the first order term in S.

$$f^*(\vec{w}_1;\vec{S})f^*(\vec{w}_2;\vec{S}) = \frac{1}{\pi^3}e^{-(w_1^2+w_2^2)}\{1+2\vec{S}.(\vec{w}_1+\vec{w}_2)\} \; . \tag{4-29}$$

The zeroth order term in (4-29) does not contribute to the drag coefficient.

We use the coordinate systems introduced in the preceding section. The polar and azimuthal angle of any vector $\vec{a}$ are denoted by θ_a, ϕ_a, θ'_a, ϕ'_a and θ''_a, ϕ''_a in the XYZ, X'Y'Z' and X''Y''Z'' system, respectively. It is also convenient to introduce the polar and azimuthal angles $\tilde{\theta}_\sigma$ and $\tilde{\phi}_\sigma$ of θ_{12} in a coordinate system with the Z-axis in the direction of $\vec{w}'_{12}$ in (4-12a) and in the direction of $\vec{w}_{12}$ in (4-12b) and the X-axis in the plane through S and $\vec{w}'_{12}$ or $\vec{w}_{12}$. We transform again the variable τ_1^* into the distance r in accordance with (4-26). In the collision integral (4-11a) we can integrate readily over the velocity $\vec{w}_1$. We thus obtain

$$\lim_{S\to 0} C_{H1}(S) = -\frac{4\sqrt{2}}{\pi^3 S}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\infty dw'_1 w'^2_1 e^{-w'^2_1}\int_0^\pi d\theta_{R1}\sin\theta_{R1}$$

$$\cdot\int_0^{\pi/2} d\theta''_{w1}\sin\theta''_{w1}\cos\theta''_{w1}\int_0^{2\pi} d\phi''_{w1}\int_1^\infty dr(1-r^{-2}\sin^2\theta''_{w1})^{-1/2}\int_{\pi-\arcsin r^{-1}}^{\pi} d\theta''_{w2}\sin\theta''_{w2}\int_0^{2\pi} d\phi''_{w2}$$

$$\cdot w'_{12}\left(\frac{\sqrt{\pi}}{2}R_{2z}-w_{2z}\right)\left(\frac{\sqrt{\pi}}{2}\cos\theta_{R1}-w_{2z}\right), \tag{4-30a}$$

$$\lim_{S\to 0} C_{H2}(S) = +\frac{4\sqrt{2}}{\pi^3 S}\int_0^\infty dw_1 w_1^2 e^{-w_1^2}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\pi d\theta_{R1}\sin\theta_{R1}$$

$$\cdot\int_0^{\pi/2} d\theta'_{w1}\sin\theta'_{w1}\cos\theta'_{w1}\int_0^{2\pi} d\phi'_{w1}\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1})^{-1/2}\int_{\pi-\arcsin r^{-1}}^{\pi} d\theta''_{w2}\sin\theta''_{w2}\int_0^{2\pi} d\phi''_{w2}$$

$$\cdot w_{12}\left(w_{2z}-\frac{\sqrt{\pi}}{2}R_{2z}\right)(w_{1z}+w_{2z}), \tag{4-30b}$$

$$\lim_{S\to 0} C_{R1}(S) = + \frac{4\sqrt{2}}{\pi^3}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\infty dw_1' w_1'^2 e^{-w_1'^2}\int_0^\pi d\theta_{R1}\sin\theta_{R1}\int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\phi_2$$

$$\cdot\int_0^{\pi/2} d\theta'_{w1'}\sin\theta'_{w1}\cos\theta'_{w1'}\int_0^{2\pi} d\phi'_{w1'}\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1'})^{-1/2}\int_{\pi/2}^{\pi} d\tilde\theta_\sigma \sin\tilde\theta_\sigma\,|\cos\tilde\theta_\sigma|\int_0^{2\pi} d\tilde\phi_\sigma$$

$$\cdot w'_{12}\left(\frac{\sqrt{\pi}}{2}R_{2z}-w''_{1z}\right)\left(\frac{\sqrt{\pi}}{2}\cos\theta_{R1}-w_2\cos\theta_2\right)\,\Theta(-\cos\theta''_{w1''})\Theta(1-r\sin\theta''_{w1''})\,, \tag{4-30c}$$

$$\lim_{S\to 0} C_{R2}(S) = - \frac{4\sqrt{2}}{\pi^3}\int_0^\infty dw_1 w_1^2 e^{-w_1^2}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\pi d\theta_{R1}\sin\theta_{R1}\int_0^\pi d\theta_2 \sin\theta_2 \int_0^{2\pi} d\phi_2$$

$$\cdot\int_0^{\pi/2} d\theta'_{w1}\sin\theta'_{w1}\cos\theta'_{w1}\int_0^{2\pi} d\phi'_{w1}\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1})^{-1/2}\int_{\pi/2}^{\pi} d\tilde\theta_\sigma \sin\tilde\theta_\sigma\,|\cos\tilde\theta_\sigma|\int_0^{2\pi} d\tilde\phi_\sigma$$

$$\cdot w_{12}\left(w''_{1z}-\frac{\sqrt{\pi}}{2}R_{2z}\right)(w_{1z}+w_{2z})\Theta(-\cos\theta''_{w1''})\Theta(1-r\sin\theta''_{w1''})\,. \tag{4-30d}$$

These integrals were again calculated numerically. The results are presented in Table VI.

4.6 Drag Coefficient C_1 of a Sphere at Arbitrary Speed Ratios

For arbitrary values of the speed ratio S we need to substitute the complete Maxwell distribution (3-6) into the collision integrals (4-11) and (4-12). In formulating the collision integrals we use the same notation as in the preceding section and obtain

$$C_{H1}(S) = - \frac{4\sqrt{2}}{\pi^4 S^2}\int_0^\infty dw_1 w_1^3 e^{-w_1^2}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\infty dw_1' w_1'^2 e^{-w_1'^2}\int_0^\pi d\theta_{R1}\sin\theta_{R1}$$

$$\cdot\int_{\pi/2}^{\pi} d\theta'_{w1}\sin\theta'_{w1}|\cos\theta'_{w1}|\int_0^{2\pi} d\phi'_{w1}\int_0^{\pi/2} d\theta'_{w1'}\sin\theta'_{w1'}\cos\theta'_{w1'}\int_0^{2\pi} d\phi'_{w1'}\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1'})^{-1/2}$$

$$\cdot\int_{\pi-\arcsin r^{-1}}^{\pi} d\theta''_{w2}\sin\theta''_{w2}\int_0^{2\pi} d\phi''_{w2} e^{-2S(S+w_{1z}+w_{2z})} w'_{12}\left(\frac{\sqrt{\pi}}{2}R_{2z}-w_{2z}\right)\,, \tag{4-31a}$$

TABLE VI

Collision integrals for the drag coefficient C_1 of a sphere in the low velocity limit.

$$\left\{\begin{array}{l} \lim_{S\to 0} C_{H1}(S) = -(2.585\pm0.014)S^{-1} \\ \lim_{S\to 0} C_{H2}(S) = +(0.299\pm0.008)S^{-1} \\ \lim_{S\to 0} C_H(S) = \lim_{S\to 0}\{C_{H1}(S)+C_{H2}(S)\} = -(2.29\pm0.01)S^{-1} \end{array}\right.$$

$$\left\{\begin{array}{l} \lim_{S\to 0} C_{R1}(S) = +(0.59\pm0.03)S^{-1} \\ \lim_{S\to 0} C_{R2}(S) = -(0.059\pm0.007)S^{-1} \\ \lim_{S\to 0} C_R(S) = \lim_{S\to 0} \{C_{R1}(S)+C_{R2}(S)\} = +(0.53\pm0.03)S^{-1} \end{array}\right.$$

$$\lim_{S\to 0} C_1(S) = \lim_{S\to 0}\{C_H(S) + 2C_R(S)\} = -(1.23\pm0.05)S^{-1}$$

$$C_{H2}(S) = +\frac{2\sqrt{2}}{\pi^3 S^2}\int_0^\infty dw_1 w_1^2 e^{-w_1^2}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\pi d\theta_{R1}\sin\theta_{R1}$$

$$\cdot\int_0^{\pi/2} d\theta'_{w1}\sin\theta'_{w1}\cos\theta'_{w1}\int_0^{2\pi} d\phi'_{w1}\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1})^{-1/2}\int_{\pi-\arcsin r^{-1}}^{\pi} d\theta''_{w2}\sin\theta''_{w2}\int_0^{2\pi} d\phi''_{w2}$$

$$\cdot\, e^{-2S(S+w_{1z}+w_{2z})}w_{12}\left(\frac{\sqrt{\pi}}{2}R_{2z} - w_{2z}\right), \qquad (4\text{-}31b)$$

$$C_{R1}(S) = +\frac{4\sqrt{2}}{\pi^5 S^2}\int_0^\infty dw_1 w_1^3 e^{-w_1^2}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\infty dw'_1 w'^2_1 e^{-w'^2_1}\int_0^\pi d\theta_{R1}\sin\theta_{R1}$$

$$\cdot\int_{\pi/2}^{\pi} d\theta'_{w1}\sin\theta'_{w1}|\cos\theta'_{w1}|\int_0^{2\pi} d\phi'_{w1}\int_0^\pi d\theta_2\sin\theta_2\int_0^{2\pi} d\phi_2\int_0^{\pi/2} d\theta'_{w1'}\sin\theta'_{w1'}\cos\theta'_{w1'}\int_0^{2\pi} d\phi'_{w1}$$

$$\cdot\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1'})^{-1/2}\int_{\pi/2}^{\pi} d\tilde{\theta}_\sigma\sin\tilde{\theta}_\sigma|\cos\tilde{\theta}_\sigma|\int_0^{2\pi} d\tilde{\phi}_\sigma e^{-2S(S+w_{1z}+w_{2z})}w'_{12}\left(\frac{\sqrt{\pi}}{2}R_{2z}-w''_{1z}\right)$$

$$\cdot\,\Theta(-\cos\theta''_{w1''})\Theta(1-r\sin\theta''_{w1''}), \qquad (4\text{-}31c)$$

$$C_{R2}(S) = -\frac{2\sqrt{2}}{\pi^4 S^2}\int_0^\infty dw_1 w_1^2 e^{-w_1^2}\int_0^\infty dw_2 w_2^2 e^{-w_2^2}\int_0^\pi d\theta_{R1}\sin\theta_{R1}$$

$$\cdot\int_0^{\pi/2} d\theta'_{w1}\sin\theta'_{w1}\cos\theta'_{w1}\int_0^{2\pi} d\phi'_{w1}\int_0^\pi d\theta_2\sin\theta_2\int_0^{2\pi} d\phi_{w2}\int_1^\infty dr(1-r^{-2}\sin^2\theta'_{w1})^{-1/2}$$

$$\cdot\int_{\pi/2}^{\pi} d\tilde{\theta}_\sigma\sin\tilde{\theta}_\sigma|\cos\tilde{\theta}_\sigma|\int_0^{2\pi} d\tilde{\phi}_\sigma e^{-2S(S+w_{1z}+w_{2z})}w_{12}\left(\frac{\sqrt{\pi}}{2}R_{2z}-w''_{1z}\right)\Theta(-\cos\theta''_{w1''})\Theta(1-r\sin\theta''_{w1''}).$$

(4-31d)

The various integrals were again computed numerically. In Table VII we present the estimated values of C_1, as well as the individual contributions C_{H1}, C_{H2}, C_{R1}, C_{R2} for $S \lesssim 3$. Just as in Section 3.6 when calculating the drag

TABLE VII

Collision integrals for the drag coefficient C_1 of a sphere ($S \leq 3$)

S	$C_{H1} \times S$	$C_{H2} \times S$	$C_{R1} \times S$	$C_{R2} \times S$	$C_1 \times S$
0.01					-1.22 ± 0.10
0.05	-2.51 ± 0.07	$+0.26 \pm 0.07$	$+0.45 \pm 0.03$	-0.09 ± 0.03	-1.16 ± 0.06
0.10					-1.24 ± 0.10
0.50	-2.94 ± 0.16	$+0.30 \pm 0.09$	$+0.65 \pm 0.05$	-0.07 ± 0.02	-1.47 ± 0.16
1.00	-4.51 ± 0.12	$+0.14 \pm 0.01$	$+1.04 \pm 0.05$	-0.045 ± 0.006	-2.36 ± 0.14
1.50	-6.47 ± 0.25	$+0.054 \pm 0.005$	$+1.48 \pm 0.07$	-0.016 ± 0.002	-3.48 ± 0.27
3.00	-17.95 ± 0.39	$+0.002 \pm 0.001$	$+5.47 \pm 0.39$	-0.0008 ± 0.0004	-7.00 ± 0.85

TABLE VIII

Collision integrals for the drag coefficient C_1 of a sphere ($S \geq 5$).

S	C_{H1}/S	C_{R1}/S	C_1/S
5.0	-1.64 ± 0.04	$+0.52 \pm 0.02$	-0.60 ± 0.05
7.5	-1.49 ± 0.03	$+0.51 \pm 0.01$	-0.49 ± 0.04
10.	-1.44 ± 0.03	$+0.50 \pm 0.01$	-0.43 ± 0.03
15.	-1.32 ± 0.03	$+0.52 \pm 0.02$	-0.37 ± 0.04
20.	-1.29 ± 0.03	$+0.52 \pm 0.02$	-0.25 ± 0.04
30.	-1.26 ± 0.02	$+0.49 \pm 0.01$	-0.29 ± 0.03
50.	-1.26 ± 0.05	$+0.50 \pm 0.01$	-0.26 ± 0.03

of a disc, we again notice that the contributions C_{H2} and C_{R2} may be neglected for speed ratios S>2. The results obtained for the larger speed ratios are presented in Table VIII. At low and high speed ratios the results approach the limiting values established earlier in Sections 4.4 and 4.5.

4.7 Discussion of Results for the Drag of a Sphere.

In summarizing our results for the drag of a sphere in the nearly free molecular flow regime, we write the drag coefficient C_D again as in (3-51)

$$C_D = C_o \left[1 + \frac{C_1}{C_o} K^{-1}\right] = C_o \left[1 + \left(\frac{C_H + 2C_R}{C_o}\right) K^{-1}\right] , \tag{4-32}$$

with

$$K^{-1} = \sqrt{2}\, \pi n \sigma^2 R. \tag{4-33}$$

A summary of the values of the coefficients in the expansion (4-32) for the drag coefficient C_D is presented in Table IX. The coefficients C_o and C_1/C_o are plotted as a function of the speed ratio S in Figs. 13 and 14 using the same scale as used earlier for the drag coefficient of a disc in Figs. 10 and 11. On comparing Table IX with Table V and Figs. 13 and 14 with Figs. 10 and 11, we note that the drag coefficient of a disc and a sphere are rather similar functions of the speed ratio. Again at low speed ratios C_1/C_o becomes independent of the speed ratio, while at large speed ratios C_1/C_o becomes proportional to the speed ratio S.

The drag exerted on a sphere in the nearly free molecular flow regime has been studied by a number of authors. Most of these studies are concerned either with low velocities S<<1 or large velocities S>>1 and we shall discuss the two cases separately.

Our results may be interpreted as the solution of the Boltzmann equation for a gas of hard spheres in the presence of the object. A study of the sphere drag based on the Boltzmann equation for Maxwellian molecules, i.e.

TABLE IX

Drag coefficient of a sphere in the nearly free molecular flow regime as a function of the speed ratio S.

$$C_D = C_o\left[1 + \frac{C_1}{C_o} K^{-1}\right] = C_o\left[1 + \left(\frac{C_H + 2C_R}{C_o}\right)K^{-1}\right]; \quad K^{-1} = \sqrt{2}\pi n\sigma^2 R.$$

S	C_o	C_H/C_o	C_R/C_o	C_1/C_o
0	$4.19S^{-1}$	-0.546±0.003	+0.126±0.006	-0.29±0.01
0.01	$4.19S^{-1}$	-0.527±0.007	+0.12±0.01	-0.29±0.02
0.05	$4.19S^{-1}$	-0.539±0.003	+0.13±0.01	-0.28±0.01
0.10	$4.20S^{-1}$	-0.529±0.008	+0.13±0.01	-0.28±0.01
0.50	$4.34S^{-1}$	-0.61±0.03	+0.13±0.01	-0.34±0.04
1.0	$4.75S^{-1}$	-0.92±0.02	+0.21±0.01	-0.50±0.03
1.0	4.75	-(0.92±0.02)S	+(0.21±0.01)S	-(0.50±0.03)S
1.5	3.58	-(0.80±0.03)S	+(0.18±0.01)S	-(0.43±0.03)S
3.0	2.61	-(0.76±0.02)S	+(0.23±0.02)S	-(0.30±0.04)S
5.0	2.32	-(0.71±0.02)S	+(0.22±0.01)S	-(0.26±0.03)S
7.5	2.19	-(0.68±0.01)S	+(0.230±0.005)S	-(0.22±0.02)S
10.	2.14	-(0.67±0.01)S	+(0.235±0.006)S	-(0.20±0.02)S
15.	2.09	-(0.63±0.01)S	+(0.248±0.008)S	-(0.13±0.02)S
20.	2.06	-(0.63±0.01)S	+(0.253±0.008)S	-(0.12±0.02)S
30.	2.04	-(0.63±0.01)S	+(0.24±0.01)S	-(0.14±0.01)S
50.	2.02	-(0.62±0.02)S	+(0.247±0.008)S	-(0.13±0.01)S
∞	2.00	-0.627S	+(0.255±0.001)S	-(0.117±0.002)S

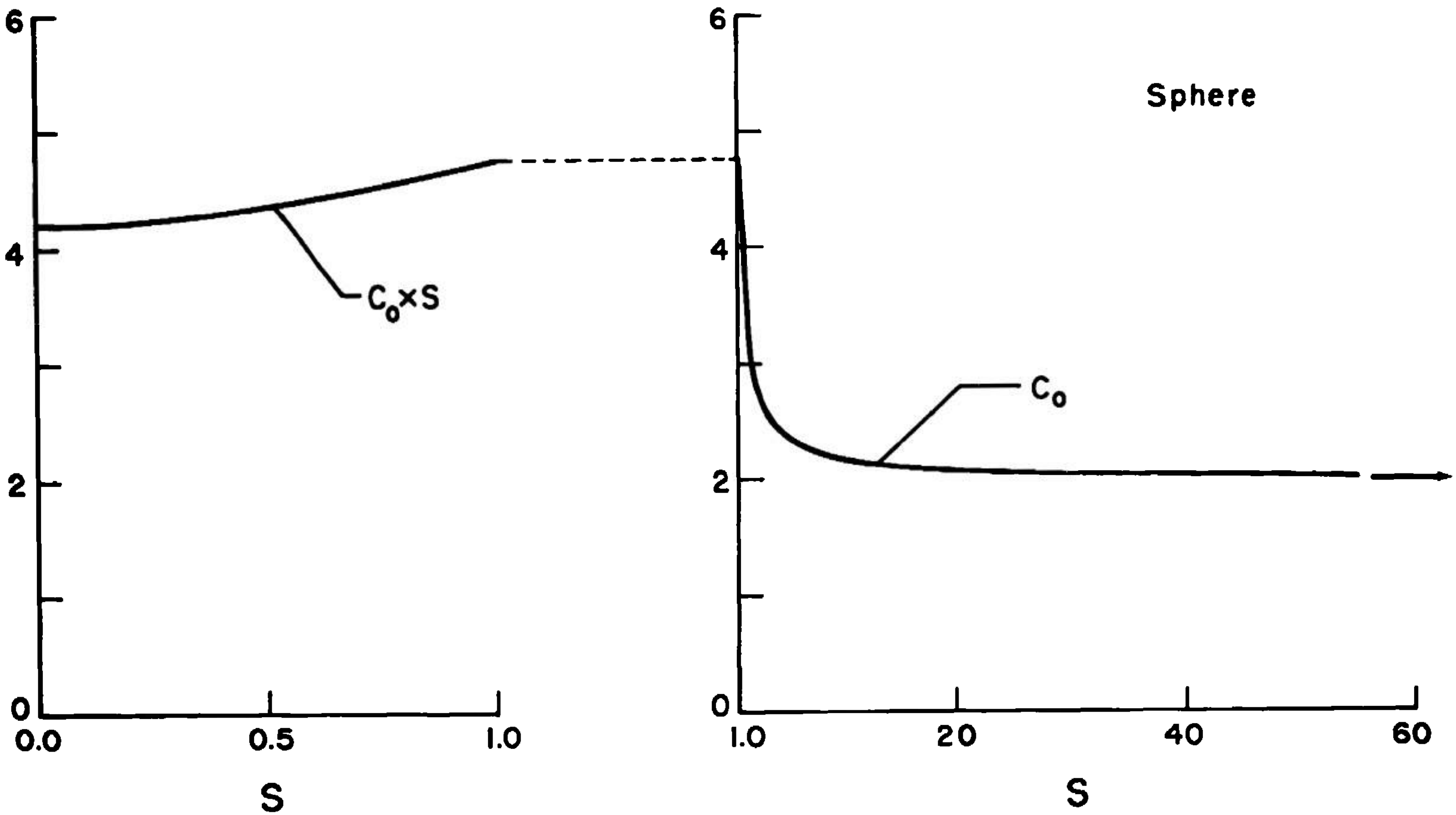


Figure 13. The drag coefficient C_o of a sphere in the free molecular flow regime as a function of the speed ratio S.

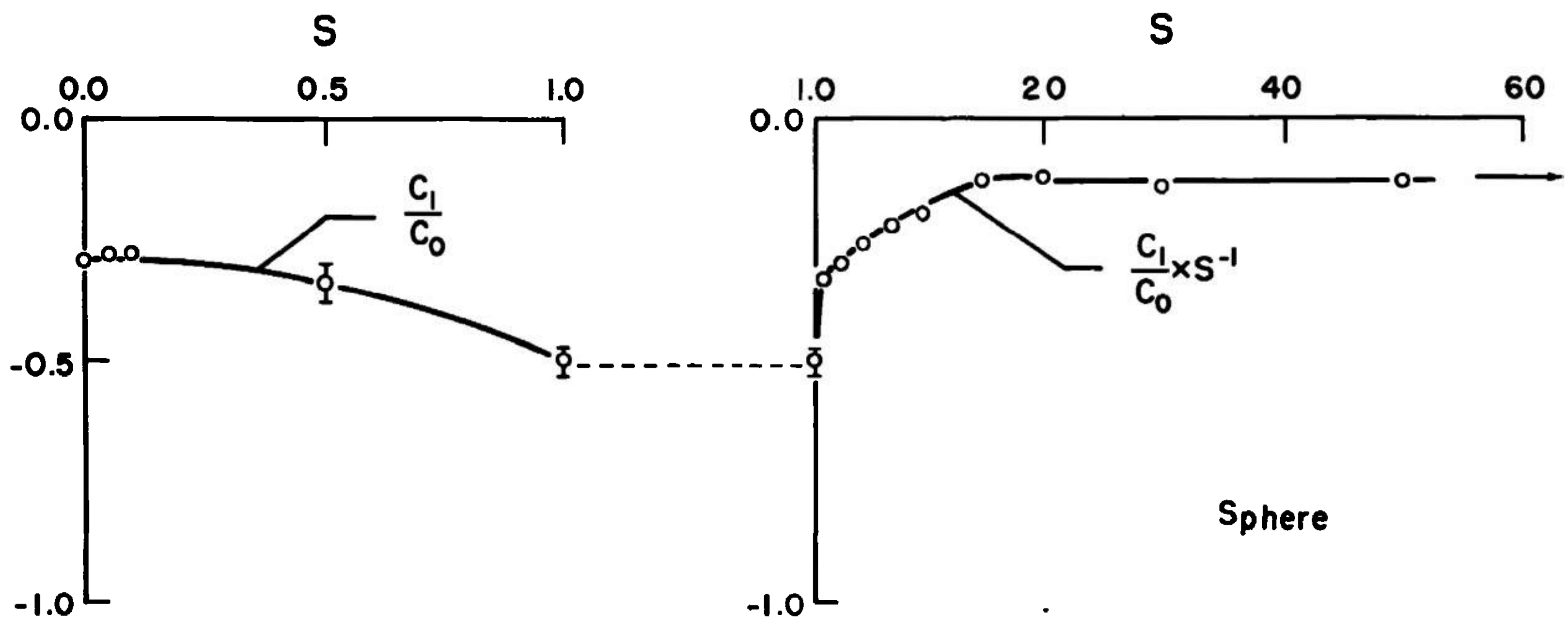


Figure 14. The coefficient C_1/C_0 of the first inverse Knudsen number correction to the drag coefficient of a sphere as a function of the speed ratio S.

molecules that repel each other with forces proportional to the inverse fifth power of the intermolecular separation, was made by Liu, Pang and Jew [4]. They expanded the collision integrals into Hermite polynomials in terms of the molecular velocity; this procedure leads to an expansion of the drag force in which higher order terms of the speed ratio S are neglected.

A comparison of our results with those of Liu et al. is made in Table X. The predicted drag coefficients appear to be very similar for $S<0.5$; in this range we confirm the conclusion of Liu et al. that the coefficient C_1/C_0 is almost independent of the speed ratio S. For $S>0.5$ the results of Liu et al. begin to deviate substantially from our results; this difference might be due to a failure of the expansion procedure of Liu et al. for larger values of S [4]. The convergence of the expansion procedure of Liu et al. has been questioned by Willis [7]. Nevertheless, unless the agreement is fortuitous, the data of Table X would suggest that the drag coefficient is insensitive to the details of the molecular interaction and that it is mainly a function of the size of the interaction range, or, equivalently, of the magnitude of the mean free path.

Due to the complications associated with solving the full Boltzmann equation many authors have used instead the BGK equation (1-4) or more sophisticated versions of this model equation. The use of this model equation introduces an uncontrolled approximation for mathematical convenience. Its predictive power is further limited by the appearance of an adjustable collision frequency ν.

In the limit $S \to o$ it is sufficient to consider a linearized version of the BGK equation (1-4). Using this procedure and assuming again diffusive

TABLE X

Comparison of our results for the sphere drag with those of Liu et al.

	Liu et al [4] (Maxwellian molecules)	This work (hard spheres)
S= 0	C_1/C_o= -0.298	C_1/C_o= -0.29±0.01
0.01	-0.298	-0.29±0.02
0.10	-0.300	-0.28±0.01
0.50	-0.308	-0.34±0.04
1.00	-0.296	-0.50±0.03

TABLE XI

Survey of theoretical values reported for $\lim_{S\to\infty} C_1/C_o$ of a sphere.

$$\lim_{S\to\infty} \frac{C_1}{C_o} = -0.24\ S$$, Baker and Charwat [14,38]

$$= -0.143\ S$$, Perepukhov [15]

$$= -0.33$$, Rose [8,39]

$$= -0.165$$, Willis [38]

$$= -(0.117\pm0.002)\ S$$, this work.

reflection of the molecules from the object, Willis calculated the coefficient C_1/C_o for a sphere with the result [7]

$$\lim_{S\to 0} \frac{C_1}{C_o} = -0.366. \tag{4-34a}$$

Starting from the same equation we find†

$$\lim_{S\to 0} \frac{C_1}{C_o} = -(0.401\pm 0.004)\,. \tag{4-34b}$$

These theoretical values are usually compared with the measurements of Millikan [36] for the drag of oil droplets in air as a function of the Knudsen number. From the experimental data of Millikan one may deduce††

$$\frac{C_1}{C_o} = -\ 0.39\pm 0.02\ . \tag{4-35}$$

The rather close agreement between (4-34) and (4-35) has been widely interpreted as a justification for the use of the BGK equation. However, such a conclusion is dangerous, since it heavily depends on the presupposition that the theoretical boundary conditions are satisfied in Millikan's experiment. In fact, Millikan's experiments were conducted in air which is a mixture, while the results are interpreted in terms of equations for a one component system. A more reliable criterion for the adequacy of the BGK equation is obtained by comparing its solution with that of the Boltzmann

---

† This work will be included in a Ph.D. thesis to be submitted by Y. Y. Lin Wang.

†† A statistical analysis of Millikan's data in terms of the theoretically predicted equation $C_D = C_o + C_1K^{-1} + C_2'K^{-2}\ln K^{-1} + C_2K^{-2}$ was made by Zondlo [37].

equation with the same boundary conditions [12]. The fact that the BGK result (4-35) differs appreciably from the Boltzmann equation result both for hard spheres and Maxwellian molecules raises doubt about the adequacy of the BGK model equation.

Theoretical calculations of the sphere drag at large velocities have been presented by Baker and Charwat [14], by Perepukhov [15], by Willis and coworkers [38] and by Rose [8,39]. A summary of the values reported for $\lim_{S\to\infty} C_1/C_0$ is presented in Table XI. In order to simplify a comparison with our results we have only quoted in Table XI the limiting values, when the temperature of the object is identified with the temperature of the gas stream. Baker and Charwat [14] and Perepukhov [15] start from the same model as considered in this report, but then proceed to introduce a number of approximations. The extent to which their results differ from our value of $-(0.117\pm0.002)S$ indicates the effect of their approximations. In particular the drastic approximations introduced by Baker and Charwat, do not appear to be justified. The fact that C_1/C_0 becomes proportional to the speed ratio S at high velocity, is a consequence of the general nature of the collision integrals.

The value obtained by Rose [8,39] is based on the BGK equation (1-4) and that obtained by Willis [38] is based on a modified version of the BGK equation. The dependence of their solutions on the speed ratio S appears to be qualitatively different. These model equation results are subject to the limitations mentioned earlier in the discussion of the drag coefficient in the low velocity limit. Their validity can only be judged a posteriori from a comparison with the solution of the Boltzmann equation, though not necessarily the solution of the Boltzmann equation for hard sphere molecules.

The fact that our calculated values of C_1/C_o become proportional to the speed ratio S at large velocities deserves some further comments. The validity of the expansion (4-32) for the drag coefficient is limited to the range where the mean free path is substantially larger than the object, i.e. the inverse Knudsen number should be substantially smaller than unity. However, the parameter $K^{-1} = \sqrt{2}\,\pi n \sigma^2 R$ in (4-32) represents the Knudsen number in the absence of (i.e. away from) the sphere and is therefore sometimes referred to as K_∞^{-1}. In order for the expansion to be valid we must require that the local inverse Knudsen number near the object is substantially smaller than unity. Indicating the local Knudsen number by $\tilde{K}$, then at small velocities

$$\lim_{S\to o} \tilde{K}^{-1} \cong K^{-1} = \sqrt{2}\,\pi n \sigma^2 R \;, \tag{4-36a}$$

while at large velocities

$$\lim_{S\to\infty} \tilde{K}^{-1} \cong SK^{-1} = S\sqrt{2}\,\pi n \sigma^2 R \;, \tag{4-36b}$$

since, at large velocities, the mean free path of the reflected molecules becomes inversely proportional to the speed ratio S. If we thus rewrite the expansion (4-32) in terms of the local Knudsen number

$$C_D = C_o\left[1 + \frac{\tilde{C}_1}{C_o}\,\tilde{K}^{-1}\right] , \tag{4-37}$$

subject to the condition

$$\tilde{K}^{-1} \ll 1, \tag{4-38}$$

then $\lim_{S\to\infty} \tilde{C}_1/C_o = -(0.117\pm0.002)$ becomes independent of the speed ratio S. Thus the apparent increase of C_1/C_o at large velocities as a function of S is a consequence of the fact that the local inverse Knudsen number itself

becomes proportional to the speed ratio S. Unfortunately, the condition (4-38) implies for large values of the speed ratio that $K^{-1} \ll S^{-1}$ and, hence, the range of validity of the expansion decreases with increasing values of the speed ratio S. Of course, this limitation applies to all theoretical results obtained by a Knudsen number iteration procedure.

Experimental data for the drag coefficient of a sphere at high velocities have been reported by Kinslow and Potter [40]. In terms of our parameters these data correspond to $SK^{-1} \approx 3$ and therefore do not overlap with the range $SK^{-1} \ll 1$ of our calculated values.

CHAPTER V

REMARKS

In this report we have considered a density expansion for the drag coefficient C_D in terms of the density of the gas stream. When the mean free path of the molecules is sufficiently large, this expansion reduces to an expansion in terms of the inverse Knudsen number. Retaining only the first two terms we thus obtained

$$C_D = C_o + C_1 K^{-1} , \tag{5-1}$$

where the parameter K^{-1} was defined as

$$K^{-1} = \sqrt{2}\,\pi n \sigma^2 R . \tag{5-2}$$

The coefficient C_o represents the drag coefficient in the limit of free molecular flow. It was shown that the coefficient C_1 is determined by a set of well defined collision integrals associated with the dynamical motion of two molecules in the presence of the object. These collision integrals can be formulated for objects of any shape. In order to demonstrate the feasibility of the method we calculated these collision integrals for the coefficient C_1 of a disc and a sphere as a function of the speed ratio S assuming that the molecules of the gas are reflected diffusively by the object. For convenience we took the temperature of the object to be equal to the temperature of the gas stream and assumed that the molecules of the gas stream could be treated as hard spheres. These approximations are not essential. Other temperatures of the object and molecules with more complicated interaction potentials can be handled by using the appropriate

binary collision operators $T(1X)$.

The range of validity of (5-1) is determined by the conditions

$$K^{-1} \ll 1 \qquad \text{for } S \lesssim 1 \,, \tag{5-3a}$$

and

$$K^{-1} \ll S^{-1} \qquad \text{for } S \gtrsim 1 \,. \tag{5-3b}$$

In order to extend this range one needs to consider higher order terms in the expansion (5-1) for the drag coefficient. In analogy to the density expansion of the transport properties of a moderately dense gas, one may anticipate an expansion of the form [5,16,17,19;41]

$$C = C_o + \tilde{C}_1 \tilde{K}^{-1} + \tilde{C}_2' \tilde{K}^{-2} \ln \tilde{K}^{-1} + \tilde{C}_2 \tilde{K}^{-2} + \ldots \tag{5-4}$$

where $\tilde{K}$ is the local Knudsen number in the neighborhood of the object. The relationship between this local Knudsen number $\tilde{K}$ and our parameter K was discussed in Section 4.7. In view of (4-36b) one therefore may anticipate higher order terms that are nonanalytic in terms of the parameter K^{-1} as well as in terms of the speed ratio or the Mach number. For objects with two dimensional geometry, such as a cylinder or a strip whose length is large compared to the mean free path, these nonanalyticities appear already in the first inverse Knudsen number correction. However, the nature of these nonanalytic terms, as well as their practical significance, is not yet well understood and further research is required.

REFERENCES

1. S. A. Schaaf, *Mechanics of Rarefied Gases*, in Handbuch der Physik, Vol. VIII/2, S. Flügge, ed. (Springer Verlag, Berlin, 1963), p. 591.

2. S. A. Schaaf and P. L. Chambre, *Flow of Rarefied Gases*, in Fundamentals of Gas Dynamics, H. W. Emmons, ed. (Princeton Univ. Press, Princeton, New Jersey, 1958), p. 687.

3. S. Chapman and T. G. Cowling, *The Mathematic Theory of Nonuniform Gases* (Cambridge Univ. Press, London and New York, 3rd ed., 1970).

4. V. C. Liu, S. C. Pang and H. Jew, Phys. Fluids 8, 788 (1965).

5. Y. P. Pao and D. R. Willis, Phys. Fluids 12, 435 (1969).

6. P. L. Bhatnagar, E. P. Gross and M. Krook, Phys. Rev. 94, 511 (1954).

7. D. R. Willis, Phys. Fluids 9, 2522 (1966).

8. M. H. Rose, Phys. Fluids 7, 1262 (1964).

9. C. Cercignani and C. D. Pagani, Phys. Fluids 11, 1395 (1968); C. Cercignani, C. D. Pagani and P Bassanini, Phys. Fluids 11, 1399 (1968).

10. M. N. Kogan, *Rarefied Gas Dynamics* (Plenum Press, New York, 1969).

11. C. Cercignani, *Mathematical Methods in Kinetic Theory* (Plenum Press, New York, 1969).

12. D. R. Willis, in *Rarefied Gas Dynamics*, Advances Applied Mechanics, Suppl. 5, Vol. I. I. L. Trilling and H. Y. Wachman, eds. (Academic Press, New York, 1969), p. 61.

13. M. Lunc and J. Lubonski, Arch. Mech. Stosowanej 8, 597 (1956); Bull. Acad. Polon. (Sec. IV) 5, 1 (1957).

14. R. M. L. Baker and A. F. Charwat, Phys. Fluids 1, 73 (1958).

15. V. A. Perepukhov, U.S.S.R. Computational Mathematics and Mathematical Physics 7(2), 276 (1967) [translated from Zh. vȳchisl. Mat. mat. Fiz. 7(2), 444 (1967)].

16. J. R. Dorfman, W. A. Kuperman, J. V. Sengers and C. F. McClure, Phys. Fluids 16, 2347 (1973).

17. C. F. McClure, *A Many Body Theory of Aerodynamic Forces*, Ph.D. Thesis (Univ. Maryland, College Park, Md., 1972); C. F. McClure and J. R. Dorfman, to be published.

18. G. E. Kelly and J. V. Sengers, *Collision Integrals for the Knudsen Number Dependence of the Growth Rate of Droplets in a Supersaturated Vapor*, Technical Report AEDC-TR-72-172 (Arnold Engineering Development Center, Tenn., 1972). See also: J. Chem. Phys. 57, 1441 (1972).

19. W. A. Kuperman, *Aerodynamic Forces on Objects in the Nearly Free Molecular Flow Regime*, Ph. D. Thesis (Univ. Maryland, College Park, Md., 1972).

20. D. R. Willis, Bull. Am. Phys. Soc. 16, 1333 (1971).

21. A. G. Keel, T. E. Chamberlain and D. R. Willis, in *Rarefied Gas Dynamics* (1974, in press).

22. J. R. Dorfman and J. V. Sengers, Bull. Am. Phys. Soc. 15, 515 (1970).

23. E. G. D. Cohen, Physica 28, 1025 (1962); J. Math. Phys. 4, 183 (1963).

24. M. S. Green and R. A. Piccirelli, Phys. Rev. 132, 1388 (1963).

25. M. H. Ernst, J. R. Dorfman, W. R. Hoegy and J. M. J. Van Leeuwen, Physica 45, 127 (1969).

26. W. R. Hoegy and J. V. Sengers, Phys. Rev. A2, 2461 (1970).

27. J. V. Sengers, M. H. Ernst and D. T. Gillespie, *Three-Particle Collision Integrals for Thermal Conductivity, Viscosity and Self-Diffusion of a Gas of Hard Spherical Molecules. Part I. Theory*, Technical Report AEDC-TR-72-142 (Arnold Engineering Development Center, Tenn., 1972). See also: J. Chem. Phys. 56, 5583 (1972).

28. J. R. Dorfman and E. G. D. Cohen, J. Math. Phys. 8, 282 (1967).

29. J. V. Sengers, Phys. Fluids 9, 1333 (1966).

30. J. V. Sengers, *Triple Collision Effects in the Thermal Conductivity and Viscosity of Moderately Dense Gases*, Technical Report AEDC-TR-69-68 (Arnold Engineering Development Center, Tenn., 1969).

31. D. T. Gillespie and J. V. Sengers, *Three-Particle Collision Integrals for Thermal Conductivity, Viscosity and Self-Diffusion of a Gas of Hard Spherical Molecules. Part II. Calculations*, Technical Report AEDC-TR-73-171 (Arnold Engineering Development Center, Tenn., 1973).

32. H. Ashley, J. Aeronaut. Sci. 16, 95 (1949).

33. D. T. Gillespie and J. V. Sengers, *Triple Collision Effects in the Thermal Conductivity and Viscosity of Moderately Dense Gases. Part II*, Technical Report AEDC-TR-71-51 (Arnold Engineering Development Center, Tenn., 1971).

34. D. T. Gillespie, *The Monte Carlo Method of Evaluating Integrals*, Technical Report (Naval Weapons Center, China Lake, Calif., to be published).

35. W. A. Kuperman, private communication.

36. R. A. Millikan, Phys. Rev. 22, 1 (1923).

37. J. W. Zondlo, *Analysis of Millikan's Measurements for the Drag of Droplets as a Function of Knudsen Number*, Master's thesis (Univ. of Maryland, College Park, Md., 1972).

38. G. J. Maslach, D. R. Willis, S. Tang and D. Ko, in *Rarefied Gas Dynamics*, Advances Applied Mechanics, Suppl. 3, Vol. I, J. H. De Leeuw, ed. (Academic Press, New York, 1965), p. 433.

39. M. H. Rose, in *Rarefied Gas Dynamics*, Advances Applied Mechanics, Suppl. 3, Vol. I, J. H. De Leeuw, ed. (Academic Press, New York, 1965), p. 312.

40. M. Kinslow and J. L. Potter, AIAA Journal 1, 2467 (1963).

41. A. L. Cooper and B. B. Hamel, Phys. Fluids 16, 35 (1973).

APPENDIX

EQUALITY OF THE CONTRIBUTIONS FROM RECOLLISIONS AND CYCLIC COLLISIONS

In this Appendix we prove that for an object in a gas stream of hard spherical molecules the recollisions and cyclic collisions yield identical contributions to the drag force, i.e.

$$\vec{E}_{C1} = \vec{E}_{R1} \quad , \tag{A-1a}$$

$$\vec{E}_{C2} = \vec{E}_{R2} \quad , \tag{A-1b}$$

where $\vec{E}_{R1}$ and $\vec{E}_{R2}$ are given by (2-65) and $\vec{E}_{C1}$ and $\vec{E}_{C2}$ by (2-66). This identity was earlier noted by Kuperman in the zero and infinite Mach number limits [19], but the theorem has a general validity independent of the speed ratio and independent of the detailed nature of the integrand. It thus also applies to other collision integrals pertaining to the nearly free molecular flow regime, such as those derived by Kelly and Sengers [18] for the mass flux to a droplet in a supersaturated vapor.

In order to prove the theorem we define the quantity

$$g(\vec{v}_1'') \equiv \int\limits_{\vec{v}_1''\cdot\hat{n}_2<0} d\vec{A}_2 \, |\vec{v}_1''\cdot\hat{n}_2| \, \delta^3(\vec{R}_1-\vec{R}_2+\vec{v}_1'\tau_1+\vec{v}_1''\tau_2) \int d\vec{v}_1'''\eta(\vec{v}_1'''|\vec{v}_1'')(\vec{v}_1'''-\vec{v}_1''), \tag{A-2}$$

and consider the integrals

$$I_R \equiv \int\limits_{\vec{v}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \, |\vec{v}_{12}'\cdot\hat{\sigma}_{12}| \, g(\vec{v}_1'') \quad , \tag{A-3a}$$

$$I_C \equiv \int\limits_{\vec{v}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12} \, |\vec{v}_{12}'\cdot\hat{\sigma}_{12}| \, g(\vec{v}_2'') \quad . \tag{A-3b}$$

On comparing (2-65a) with (2-66a) we note that $\vec{E}_{C1} = \vec{E}_{R1}$, if $I_R = I_C$.

For a gas of hard spheres the velocities before and after the second collision are interrelated by (2-18)

$$\vec{v}_1'' = \vec{v}_1' - (\vec{v}_{12}'\cdot\hat{\sigma}_{12})\hat{\sigma}_{12}\ , \tag{A-4a}$$

$$\vec{v}_2'' = \vec{v}_2 + (\vec{v}_{12}'\cdot\hat{\sigma}_{12})\hat{\sigma}_{12}\ . \tag{A-4b}$$

Let us introduce a new unit vector $\hat{\sigma}'$ defined as

$$\hat{\sigma}' \equiv -\frac{\hat{v}_{12}' - (\hat{v}_{12}'\cdot\hat{\sigma}_{12})\hat{\sigma}_{12}}{\sqrt{1-(\hat{v}_{12}'\cdot\hat{\sigma}_{12})^2}}\ , \tag{A-5}$$

so that

$$\vec{v}_2'' = \vec{v}_1' - (\vec{v}_{12}'\cdot\hat{\sigma}')\hat{\sigma}'. \tag{A-6}$$

We choose a coordinate system with the Z-axis in the direction of $\hat{v}_{12}'$. The polar and azimuthal angles of $\hat{\sigma}_{12}$ in this coordinate system are θ and ϕ and $\hat{v}_{12}'\cdot\hat{\sigma}_{12} = \cos\theta$. It follows from (A-5) that the polar and azimuthal angles θ' and ϕ' of $\hat{\sigma}'$ in the same coordinate system are $\theta' = \frac{3\pi}{2} - \theta$, $\phi' = \phi + \pi$ and that $\hat{v}_{12}'\cdot\hat{\sigma}' = \cos\theta' = -\sin\theta$. Thus

$$I_C = \int_{\vec{v}_{12}'\cdot\hat{\sigma}_{12}<0} d\hat{\sigma}_{12}\ |\vec{v}_{12}'\cdot\hat{\sigma}_{12}|\ g(\vec{v}_2'') = v_{12}'\int_0^{2\pi} d\phi \int_{\pi/2}^{\pi} d\theta\ \sin\theta\,|\cos\theta|\ g(\vec{v}_2'') =$$

$$= v_{12}'\int_0^{2\pi} d\phi' \int_{\pi/2}^{\pi} d\theta' \sin\left(\frac{3\pi}{2} - \theta'\right) \left|\cos\left(\frac{3\pi}{2} - \theta'\right)\right| . g(\vec{v}_2'') =$$

$$= v_{12}'\int_0^{2\pi} d\phi' \int_{\pi/2}^{\pi} d\theta' \sin\theta'\ |\cos\theta'|\ g(\vec{v}_2'') = \int_{\vec{v}_{12}'\cdot\hat{\sigma}'<0} d\hat{\sigma}'\ |\vec{v}_{12}'\cdot\hat{\sigma}'|\ g(\vec{v}_2''). \tag{A-7}$$

On comparing (A-7) and (A-6) with (A-3a) and (A-4a), we conclude that indeed

$$I_C = I_R \quad , \tag{A-8}$$

which implies the equality $\vec{E}_{C1} = \vec{E}_{R1}$. The equality $\vec{E}_{C2} = \vec{E}_{R2}$ follows if $\vec{v}_1'$ is identified with $\vec{v}_1$.